

# One-loop $\phi$ -MHV amplitudes using the unitarity bootstrap

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**ABSTRACT:** We consider a Higgs boson coupled to gluons via the five-dimensional effective operator  $H\text{tr}G_{\mu\nu}G^{\mu\nu}$  produced by considering the heavy top quark limit of the one-loop coupling of Higgs and gluons in the standard model. We treat  $H$  as the real part of a complex field  $\phi$  that couples to the selfdual gluon field strengths and compute the one-loop corrections to amplitudes involving  $\phi$ , two colour adjacent negative helicity gluons and an arbitrary number of positive helicity gluons - the so-called  $\phi$ -MHV amplitudes. We use four-dimensional unitarity to construct the cut-containing contributions and the recently developed recursion relations to obtain the rational contribution for an arbitrary number of external gluons. We solve the recursion relations and give explicit results for up to four external gluons. These amplitudes are relevant for Higgs plus jet production via gluon fusion in the limit where the top quark mass is large compared to all other scales in the problem.

**KEYWORDS:** Higgs Physics, NLO Computations, QCD.

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## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. The Higgs model</b>	<b>5</b>
2.1 Colour ordering	6
2.2 Tree level $\phi$ amplitudes	7
<b>3. The cut-constructible contributions</b>	<b>7</b>
3.1 Pure cut contributions	9
3.2 Cut-completion terms	13
<b>4. The rational contributions</b>	<b>14</b>
4.1 Recursive terms	16
4.2 Overlap terms	18
<b>5. The 4-point amplitude</b>	<b>21</b>
<b>6. Cross checks and limits</b>	<b>22</b>
6.1 Infra-red pole structure	22
6.2 Collinear limits	23
6.3 Collinear factorisation of the cut-constructible contributions	24
6.3.1 The mixed helicity collinear limit	25
6.3.2 Two collinear negative helicity gluons	26
6.3.3 Two collinear positive helicity gluons	26
6.3.4 Collinear factorisation of the rational contributions	27
6.4 Soft Higgs limit	28
6.4.1 Soft limit of $A_4^{(1)}(\phi, 1^-, 2^-, 3^+, 4^+)$	29
<b>7. Conclusions</b>	<b>30</b>
<b>A. Spinor conventions</b>	<b>31</b>
<b>B. Scalar integrals</b>	<b>32</b>

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## 1. Introduction

As the time for physics at CERN's Large Hadron Collider (LHC) approaches there is a great need for precision calculations of Standard Model processes. The eagerly anticipated new physics signals typically result in complex multiparticle final states that are contaminated

by Standard Model contributions. Isolating the signal therefore relies on precise theoretical calculations of background rates.

At present, the rates for such processes can be estimated relatively easily at leading order (LO) in parton level perturbation theory. A wide range of tools are available, for example ALPGEN [1, 2], the COMPHEP package [3, 4], HELEC/PHEGAS [5, 6], MADGRAPH [7, 8] and SHERPA/AMEGIC++ [9, 10] which compute the amplitudes numerically<sup>1</sup> and provide a suitable phase space over which they can be integrated. However, the predicted event rates suffer large uncertainties due to the choice of the unphysical renormalisation and factorisation scales so that the calculated rate is only an “order of magnitude” estimate. In addition, there is a rather poor mismatch between the “single parton becomes a jet” approach used in LO perturbation theory and the complicated multi-hadron jet observed in experiment.

While these problems cannot be entirely solved within perturbation theory, the situation can be improved by calculating the strong next-to-leading order (NLO) corrections. The uncertainty on the rate may be reduced to the 10%-30% level by including the next-to-leading order (NLO) corrections, which may themselves be large when new channels are accessed. In addition, the additional parton radiated into the final state allows a better modelling of the inter- and intra-jet energy flow as well as identifying regions where large logarithms must be resummed.

The ingredients necessary for computing the NLO correction to a  $n$  particle process are well known. First, one needs the tree-level contribution for the  $n + 1$  particle process where an additional parton is radiated. Second, one needs the one-loop  $n$  particle matrix elements. Both terms are infrared (and usually ultraviolet) divergent and must be carefully combined to yield an infrared and ultraviolet finite NLO prediction.

The real emission contribution is relatively well under control and can easily be automated [1–4, 7–9, 11]. The infrared singularities that occur when a parton is soft (or when two partons become collinear) can then be removed using well established (dimensional regularisation) techniques so that the “subtracted” matrix element is finite and can be evaluated in 4-dimensions [15].

The bottleneck in deriving NLO corrections for multiparticle processes is computing the one-loop amplitudes. The standard approach is to compute the relevant Feynman diagrams using a variety (or combination) of numerical and algebraic techniques. Much progress has been made in this way, numerical evaluations of processes with up to six particles have been performed [16–18]. However, one always observes large cancellations between the contributions of different Feynman diagrams and the result is generally far more compact than would naively be expected. This is a strong hint that more direct and efficient ways of performing the calculation exist. The simplicity can be realised using on-shell methods, where the cancellations due to gauge invariance and momentum conservation are already present, to compute the amplitude.

In a series of pioneering papers, Bern et al [19, 20] developed the use of on-shell methods

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<sup>1</sup>COMPHEP, MADGRAPH and SHERPA/AMEGIC++ evaluate sums of Feynman diagrams. However, both ALPGEN and HELAC/PHEGAS are based on off-shell recursion relations [11–14].

at loop level by sewing together four-dimensional tree-level amplitudes and using unitarity to reconstruct the (poly)logarithmic cut constructible part of the amplitude. A key feature of the unitarity method is that the tree-level helicity amplitudes are often very simple. However, the rational terms, that are produced when computing non-supersymmetric amplitudes in  $D = 4 - 2\epsilon$  dimensions, were difficult to obtain.

Recently, on-shell methods have received renewed attention with Witten's proposal of a duality between  $\mathcal{N} = 4$  supersymmetric Yang-Mills and a topological string theory [21]. New analytical methods, the MHV rules [22] and on-shell recursion relations [23], have been derived and have been extremely successful in computing tree-level helicity amplitudes. Both of these techniques can be proved using simple complex analysis and a knowledge of the factorisation properties of the amplitudes on multi-particle poles [24, 25]. Although developed for in the context of multigluon amplitudes, they have had wide ranging applications including processes with massive coloured scalars [26, 27], fermions [28–31], Higgs bosons [32, 33] and vector bosons [34, 28]. A more complete set of references can be found in refs. [35, 36].

Some of these on-shell ideas have been applied to one-loop amplitudes including the application of maximally-helicity-violating (MHV) vertices [37], a more generalised unitarity [38] using complex momenta and the use of the holomorphic anomaly [39]. Together with previous results [19, 20], these new methods have led to the complete analytic expressions for the cut-containing contributions [40–42] for one-loop QCD six-gluon amplitude. Very recently an efficient method for direct extraction of the integral coefficients exploiting complex momenta has also been proposed [43].

However, these techniques are also intrinsically 4-dimensional and, although providing effective ways of computing the cut-constructible (poly)logarithmic part of the amplitude, suffer from the same limitations as the older unitarity based methods and miss the rational part. These “missing” rational terms can be obtained directly from the unitarity approach by taking the cut loop momentum to be in  $D = 4 - 2\epsilon$  dimensions [44] and there have been recent developments in this direction [45–48].

On the other hand, the rational part is essentially tree-level-like in containing only poles in the complex plane. One can therefore attempt to isolate these terms using an on-shell recursion relation in an analogous way to tree level amplitudes. This is the “unitarity on-shell bootstrap” technique which combines unitarity with on-shell recursion [49–51]. Other Feynman diagram based methods have also been developed for calculating the rational terms directly [52–56]. Altogether, it has already been possible to calculate the full set of complete 6 gluon helicity amplitudes as well as closed forms for  $n$ -point amplitudes for specific helicities [57–59], and the complete set of six photon amplitudes [60, 61].

As mentioned earlier, the MHV rules have also been successfully applied to tree level QCD amplitudes involving a massive colourless scalar, the Higgs boson, in the large top mass limit [32, 33]. This is achieved using a decomposition of the Higgs field into its selfdual and anti-selfdual components,  $\phi$  and  $\phi^\dagger$ . The purpose of this paper is to apply the new on-shell methods to one-loop calculations of the Higgs plus  $n$ -gluon amplitudes. Amplitudes of this kind have been considered in the context of on-shell recursion in relations by Berger, Del Duca and Dixon [62] for the finite helicity configurations that vanish at tree level, the

$\phi$ -“all plus” and  $\phi$ -“almost all plus” amplitudes. The cut-constructible parts of the infrared divergent amplitudes involving the  $\phi$  and an arbitrary number of negative helicity gluons were studied in ref. [63]. Here we consider the amplitudes with two adjacent negative helicity gluons and any number of positive helicity gluons, the  $\phi$ -MHV amplitudes. These represent one of the ingredients needed for the complete set of helicity amplitudes for Higgs to four gluon process which has been computed using numerical methods in reference [64].

Following [51], we will split the calculation into two parts, evaluating the “pure” 4-dimensional cut-constructible  $C_n$  and rational  $R_n$  parts of the leading colour contribution to the one-loop  $n$ -point amplitude separately:

$$A_n^{(1)} = C_n + R_n. \tag{1.1}$$

For simplicity, we have dropped the leading colour subscript,  $A_{n;1}^{(1)} \equiv A_n^{(1)}$ .  $C_n$  contains all of the terms originating in box, triangle bubble loop integrals which are related to cut-containing logarithms and dilogarithms (or  $\pi^2$ ) and which are cut-constructible in 4-dimensions. In this paper we choose to compute the cut-containing contribution using the 1-loop MHV rules developed by Brandhuber, Spence and Travaglini [37]. In addition,  $C_n$  contains fake singularities that come from tensor loop integrals. To explicitly remove these spurious singularities, it is convenient to introduce a cut-completing rational term, so that the “full” cut-constructible term  $C_n$  is given by

$$\widehat{C}_n = C_n + CR_n, \tag{1.2}$$

with the corresponding modification to the rational part,

$$\widehat{R}_n = R_n - CR_n. \tag{1.3}$$

Because the rational part contains only simple poles, the aim is to construct this recursively using the multiparticle factorisation properties of amplitudes. This means constructing a direct recursive term,  $R_n^D$ , by summing over products of lower point tree and one-loop amplitudes. By construction,  $R_n^D$  encodes the complete residues on the physical poles. The cut-completion contribution  $CR_n$  may also give a contribution in the physical channels which would then lead to double counting. These potential unwanted contributions are removed by the overlap term, so that

$$\widehat{R}_n = R_n^D + O_n, \tag{1.4}$$

and the full amplitude is given by

$$A_n^{(1)} = C_n + CR_n + R_n^D + O_n. \tag{1.5}$$

Our paper is organised as follows. In section 2 we introduce the complex scalar field  $\phi$  and review the effective interaction that couples it directly to gluons. The relationship between  $\phi$  (and  $\phi^\dagger$ ) and Higgs amplitudes is spelt out, along with our conventions for colour ordering amplitudes and a selection of explicit results for tree-level  $\phi$ -amplitudes. We address the computation of the cut-constructible  $C_n$  and cut-completion  $CR_n$  parts of the one-loop

amplitude in section 3. The rational contributions are considered in section 4 where we establish the on-shell recursion relation  $R_n^D$  and give expressions for the overlap terms  $O_n$ . We solve the arbitrary multiplicity results given in sections 3 and 4 for the special case  $n = 4$  in section 5 and give an explicit analytic expression for the  $A_4^{(1)}$  amplitude. Section 6 is dedicated to a series of checks of our results. We show that our results have the correct infrared pole structure, satisfy the correct collinear limits and we study the limit where the momentum of the Higgs boson becomes soft. Finally, our findings are summarised in section 7. Two appendices are enclosed that define our spinor notation and list the relevant one-loop functions that appear in  $C_n$ .

## 2. The Higgs model

In the Standard Model the Higgs boson couples to gluons through a fermion loop. The dominant contribution is from the top quark. For large  $m_t$ , the top quark can be integrated out leading to the effective interaction [65, 66],

$$\mathcal{L}_H^{\text{int}} = \frac{C}{2} H \text{tr} G_{\mu\nu} G^{\mu\nu} . \tag{2.1}$$

In the Standard Model, the strength of the interaction  $C$  has been calculated up to order  $\mathcal{O}(\alpha_s^4)$  in [67]. However, for our purposes we need it only up to order  $\mathcal{O}(\alpha_s^2)$  [68],

$$C = \frac{\alpha_s}{6\pi v} \left( 1 + \frac{11}{4} \frac{\alpha_s}{\pi} + \dots \right) , \tag{2.2}$$

with  $v = 246$  GeV. This approximation works very well under the condition that the kinematic scales involved are smaller than twice the top quark mass  $M_t$ . [71, 69, 70]

The MHV structure of the Higgs-plus-gluons amplitudes is best elucidated [32] by considering  $H$  to be the real part of a complex field  $\phi = \frac{1}{2}(H + iA)$ , so that

$$\begin{aligned} \mathcal{L}_{H,A}^{\text{int}} &= \frac{C}{2} \left[ H \text{tr} G_{\mu\nu} G^{\mu\nu} + iA \text{tr} G_{\mu\nu} *G^{\mu\nu} \right] \\ &= C \left[ \phi \text{tr} G_{SD\mu\nu} G_{SD}^{\mu\nu} + \phi^\dagger \text{tr} G_{ASD\mu\nu} G_{ASD}^{\mu\nu} \right] \end{aligned} \tag{2.3}$$

where the purely selfdual (SD) and purely anti-selfdual (ASD) gluon field strengths are given by

$$G_{SD}^{\mu\nu} = \frac{1}{2}(G^{\mu\nu} + *G^{\mu\nu}), \quad G_{ASD}^{\mu\nu} = \frac{1}{2}(G^{\mu\nu} - *G^{\mu\nu}),$$

with

$$*G^{\mu\nu} \equiv \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma} .$$

The important observation of [32] was that, due to selfduality, the amplitudes for  $\phi$  plus  $n$  gluons, and those for  $\phi^\dagger$  plus  $n$  gluons, each have a simpler structure than the gluonic amplitudes for either  $H$  or  $A$ . Amplitudes can be constructed for  $\phi$  plus  $n$  gluons and for  $\phi^\dagger$  plus  $n$  gluons separately.

The decomposition of the  $HGG$  and  $AGG$  vertices into the self-dual and the anti-self-dual terms eq. (2.3) means that the Higgs and pseudoscalar Higgs amplitudes are obtained from  $\phi$  and  $\phi^\dagger$  amplitudes.

$$\mathcal{A}_n^{(m)}(H, g_1^{\lambda_1}, \dots, g_n^{\lambda_n}) = \mathcal{A}_n^{(m)}(\phi, g_1^{\lambda_1}, \dots, g_n^{\lambda_n}) + \mathcal{A}_n^{(m)}(\phi^\dagger, g_1^{\lambda_1}, \dots, g_n^{\lambda_n}), \quad (2.4)$$

$$\mathcal{A}_n^{(m)}(A, g_1^{\lambda_1}, \dots, g_n^{\lambda_n}) = \mathcal{A}_n^{(m)}(\phi, g_1^{\lambda_1}, \dots, g_n^{\lambda_n}) - \mathcal{A}_n^{(m)}(\phi^\dagger, g_1^{\lambda_1}, \dots, g_n^{\lambda_n}). \quad (2.5)$$

However, parity further relates  $\phi$  and  $\phi^\dagger$  amplitudes,

$$\mathcal{A}_n^{(m)}(\phi^\dagger, g_1^{\lambda_1}, \dots, g_n^{\lambda_n}) = \left( \mathcal{A}_n^{(m)}(\phi, g_1^{-\lambda_1}, \dots, g_n^{-\lambda_n}) \right)^*. \quad (2.6)$$

From now on, we will only consider  $\phi$  amplitudes, knowing that all others can be obtained using eqs. (2.4)–(2.6).

## 2.1 Colour ordering

The tree level amplitudes for a  $\phi$  and  $n$  gluons can be decomposed into colour ordered amplitudes as [73, 72],

$$\mathcal{A}_n^{(0)}(\phi, \{k_i, \lambda_i, a_i\}) = iCg^{n-2} \sum_{\sigma \in S_n/Z_n} \text{tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) A_n^{(0)}(\phi, \sigma(1^{\lambda_1}, \dots, n^{\lambda_n})). \quad (2.7)$$

Here  $S_n/Z_n$  is the group of non-cyclic permutations on  $n$  symbols, and  $j^{\lambda_j}$  labels the momentum  $p_j$  and helicity  $\lambda_j$  of the  $j^{\text{th}}$  gluon, which carries the adjoint representation index  $a_i$ . The  $T^{a_i}$  are fundamental representation  $SU(N_c)$  color matrices, normalized so that  $\text{Tr}(T^a T^b) = \delta^{ab}$ . The strong coupling constant is  $\alpha_s = g^2/(4\pi)$ .

Tree-level amplitudes with a single quark-antiquark pair can be decomposed into colour-ordered amplitudes as follows,

$$\begin{aligned} & \mathcal{A}_n(\phi, \{p_i, \lambda_i, a_i\}, \{p_j, \lambda_j, i_j\}) \\ &= iCg^{n-2} \sum_{\sigma \in S_{n-2}} (T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n-1)}})_{i_1 i_n} A_n(\phi, 1^\lambda, \sigma(2^{\lambda_2}, \dots, (n-1)^{\lambda_{n-1}}), n^{-\lambda}). \end{aligned} \quad (2.8)$$

where  $S_{n-2}$  is the set of permutations of  $(n-2)$  gluons. Quarks are characterised with fundamental colour label  $i_j$  and helicity  $\lambda_j$  for  $j = 1, n$ . By current conservation, the quark and antiquark helicities are related such that  $\lambda_1 = -\lambda_n \equiv \lambda$  where  $\lambda = \pm \frac{1}{2}$ .

The 1-loop gluonic amplitudes which are the main subject of this paper follow the same colour ordering as the pure QCD amplitudes [19] and can be decomposed as [62, 63],

$$\mathcal{A}_n^{(1)}(\phi, \{k_i, \lambda_i, a_i\}) = iCg^n \sum_{c=1}^{[n/2]+1} \sum_{\sigma \in S_n/S_{n;c}} G_{n;c}(\sigma) A_n^{(1)}(\phi, \sigma(1^{\lambda_1}, \dots, n^{\lambda_n})) \quad (2.9)$$

where

$$G_{n;1}(1) = N \text{tr}(T^{a_1} \dots T^{a_n}) \quad (2.10)$$

$$G_{n;c}(1) = \text{tr}(T^{a_1} \dots T^{a_{c-1}}) \text{tr}(T^{a_c} \dots T^{a_n}), \quad c > 2. \quad (2.11)$$

The sub-leading terms can be computed by summing over various permutations of the leading colour amplitudes [19].

## 2.2 Tree level $\phi$ amplitudes

As noted in [32] the all-plus and almost all-plus  $\phi$  amplitudes vanish,

$$A_n^{(0)}(\phi, g_1^+, g_2^+, g_3^+, \dots, g_n^+) = 0, \quad (2.12)$$

$$A_n^{(0)}(\phi, g_1^-, g_2^+, g_3^+, \dots, g_n^+) = 0, \quad (2.13)$$

for all  $n$ .

The tree  $\phi$ -amplitudes, with precisely two negative helicities are the first non-vanishing  $\phi$  amplitudes. These amplitudes are the  $\phi$ -MHV amplitudes and general factorisation properties now imply that they have to be extremely simple [32]. For the case when legs  $q$  and  $p$  have negative helicity, they are given by

$$A_n^{(0)}(\phi, g_1^+, g_2^+, \dots, g_p^-, \dots, g_q^-, \dots, g_n^+) = \frac{\langle p q \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \dots \langle n-1, n \rangle \langle n 1 \rangle}, \quad (2.14)$$

In fact, the expression eq. (2.14) for the  $\phi$ -MHV  $n$ -gluon amplitude has precisely the same form as the MHV  $n$ -gluon amplitudes in pure QCD [74]. The only difference is that the total momentum carried by gluons,  $p_1 + p_2 + \dots + p_n = -p_\phi$  is the momentum carried by the  $\phi$ -field and is non-zero.

There are two  $\phi$ -MHV amplitudes involving a quark pair,

$$A_n(\phi, q_1^-, \dots, g_r^-, \dots, \bar{q}_n^+) = \frac{\langle r 1 \rangle^3 \langle r n \rangle}{\langle 1 2 \rangle \langle 2 3 \rangle \dots \langle n-1, n \rangle \langle n 1 \rangle}, \quad (2.15)$$

$$A_n(\phi, q_1^+, \dots, g_r^-, \dots, \bar{q}_n^-) = \frac{\langle r 1 \rangle \langle r n \rangle^3}{\langle 1 2 \rangle \langle 2 3 \rangle \dots \langle n-1, n \rangle \langle n 1 \rangle}. \quad (2.16)$$

We note in passing that the tree  $\phi$ -amplitude with all negative helicity gluons, the  $\phi$ -all-minus amplitude, also has a simple structure [32],

$$A_n^{(0)}(\phi; 1^-, \dots, n^-) = (-1)^n \frac{m_H^4}{[1 2] [2 3] \dots [n-1, n] [n 1]}. \quad (2.17)$$

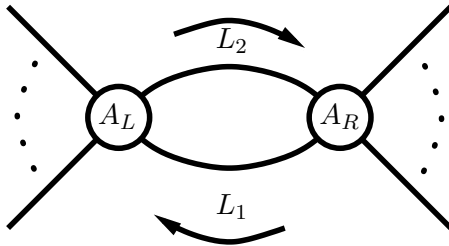
Amplitudes with fewer (but more than two) negative helicities have been computed with Feynman diagrams (up to 4 partons) in ref. [72] and using MHV rules and on-shell recursion relations in refs. [32, 33].

The main goal of this paper is the construction of the one-loop  $\phi$ -MHV amplitude with two adjacent negative helicities. For definiteness, we focus on the specific helicity configuration  $(1^-, 2^-, 3^+, \dots, n^+)$ .

## 3. The cut-constructible contributions

In a landmark paper, Brandhuber, Spence and Travaglini [37] showed that it is possible to calculate one-loop MHV amplitudes in  $\mathcal{N} = 4$  using MHV rules. The calculation has many similarities to the unitarity based approach of refs. [19, 20], the main difference being that the MHV rules reproduce the cut-constructible parts of the amplitude directly, without having to worry about double counting. This is the method that we wish to employ here.





**Figure 1:** A generic one loop MHV diagram or unitarity cut.

The four-dimensional cut-constructible part of one-loop amplitudes can be constructed by joining two on-shell vertices by two scalar propagators, both of which need to be continued off-shell. A generic diagram is shown in figure 1 and the full amplitude will be a sum over all possible permutations and helicity configurations. In the BST approach the propagators are continued off-shell as for the tree level MHV rules and can be written [37],

$$L_i = l_i + z_i \eta. \quad (3.1)$$

Loop integration over the  $L_i$  are related to phase space integration over  $l_i$  and trivial integrals over  $z_i$ . Note that this approach actually performs all integrals and therefore there is a one-to-one identification of cut diagrams and cut-constructible functions. However, it relies on being able to perform the phase space integral. For the cases considered in this paper, the phase space integrals are known and are directly related to loop functions.

A generic diagram can be written:

$$\mathcal{D} = \frac{1}{(2\pi)^4} \int \frac{d^4 L_1}{L_1^2} \frac{d^4 L_2}{L_2^2} \delta^{(4)}(L_1 - L_2 - P) A_L(l_1, -P, -l_2) A_R(l_2, P, -l_1) \quad (3.2)$$

where  $A_{L(R)}$  are the amplitudes for the left(right) vertices and  $P$  is the sum of momenta incoming to the right hand amplitude. The important step is the evaluation of this expression is to re-write the integration measure as an integral over the on-shell degrees of freedom and a separate integral over the complex variable  $z$  [37]:

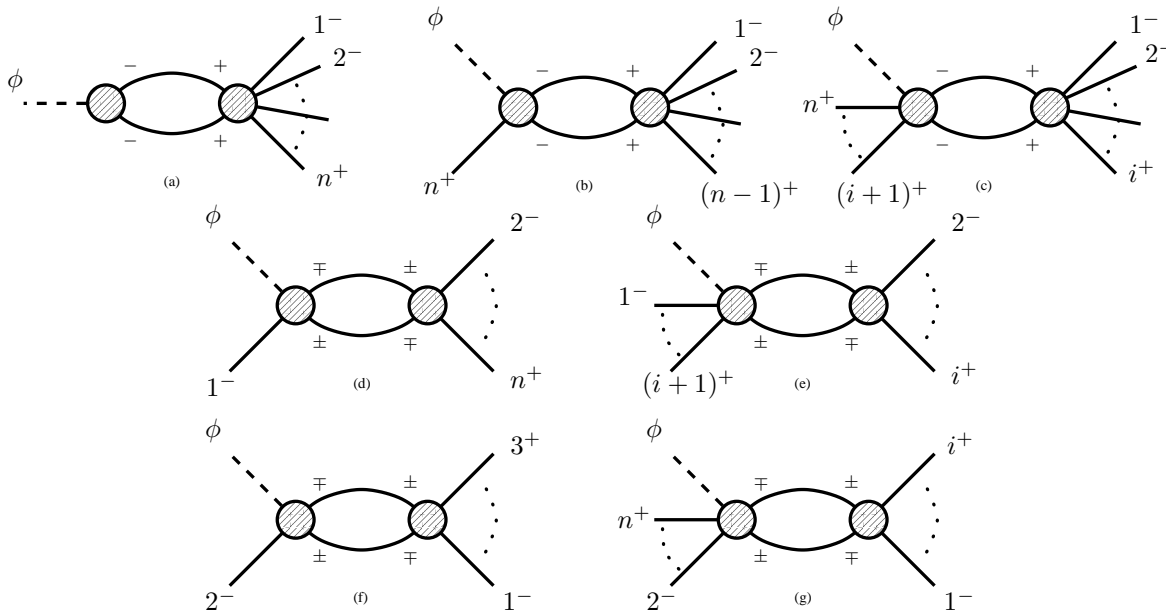
$$\begin{aligned} \frac{d^4 L_1}{L_1^2} \frac{d^4 L_2}{L_2^2} &= (4i)^2 \frac{dz_1}{z_1} \frac{dz_2}{z_2} d^4 l_1 d^4 l_2 \delta^{(+)}(l_1^2) \delta^{(+)}(l_2^2) \\ &= (4i)^2 \frac{2dzdz'}{(z-z')(z+z')} d^4 l_1 d^4 l_2 \delta^{(+)}(l_1^2) \delta^{(+)}(l_2^2), \end{aligned} \quad (3.3)$$

where  $z = z_1 - z_2$  and  $z' = z_1 + z_2$ . The integrand can only depend on  $z, z'$  through the momentum conserving delta function,

$$\delta^{(4)}(L_1 - L_2 - P) = \delta^{(4)}(l_1 - l_2 - P + z\eta) = \delta^{(4)}(l_1 - l_2 - \hat{P}), \quad (3.4)$$

where  $\hat{P} = P - z\eta$ . This means that the integral over  $z'$  can be performed so that,

$$\begin{aligned} \mathcal{D} &= \frac{(4i)^2 2\pi i}{(2\pi)^4} \int \frac{dz}{z} \int d^4 l_1 d^4 l_2 \delta^{(+)}(l_1^2) \delta^{(+)}(l_2^2) \delta^{(4)}(l_1 - l_2 - \hat{P}) A_L(l_1, -P, -l_2) A_R(l_2, P, -l_1) \\ &= (4i)^2 2\pi i \int \frac{dz}{z} \int d\text{LIPS}^{(4)}(-l_1, l_2, \hat{P}) A_L(l_1, -P, -l_2) A_R(l_2, P, -l_1), \end{aligned} \quad (3.5)$$



**Figure 2:** The MHV loop diagrams contributing to the  $\phi \rightarrow g_1^- g_2^- g_3^+ \cdots g_n^+$  amplitude.

where,

$$d\text{LIPS}^{(4)}(-l_1, l_2, \hat{P}) = \frac{1}{(2\pi)^4} d^4 l_1 d^4 l_2 \delta^{(+)}(l_1^2) \delta^{(+)}(l_2^2) \delta^{(4)}(l_1 - l_2 - \hat{P}) \quad (3.6)$$

The phase space integral is regulated using dimensional regularisation. Tensor integrals arising from the product of tree amplitudes can be reduced to scalar integrals either by using spinor algebra or standard Passarino-Veltman reduction. The remaining scalar integrals have been evaluated previously by van Neerven [44].

### 3.1 Pure cut contributions

The pure cut contribution is constructed by connecting two tree-level vertices and the seven independent topologies are shown in figure 2. Note that the last four topologies helicity configurations allow both fermionic and gluonic contributions. Note also that all fermion loops always appear in association with a factor of  $N_F$ , the number of fermion species, and a factor of  $-1$ .

Let us consider diagram 2(a) to begin with. We can take the momenta to be labelled from 1 to  $n$  around the right hand amplitude so that we consider a cut in the  $s_{1,n}$  channel, i.e. when the momentum flowing across the cut is  $p_1 + p_2 + \cdots + p_n$  corresponding to an invariant mass  $s_{1,n} = (p_1 + p_2 + \cdots + p_n)^2$ .<sup>2</sup> Other diagrams with this topology are accessible by permuting the arguments of the  $s_{1,n}$  channel. The product of the two vertices can be

<sup>2</sup>Note that when  $j < i$ ,  $s_{i,j} \equiv (p_i + \cdots + p_n + p_1 + \cdots + p_j)^2$ .

written:

$$\begin{aligned}
 A_L A_R &= -\frac{m_H^4}{\langle l_1 l_2 \rangle^2} \frac{\langle 12 \rangle^4}{\langle n l_1 \rangle \langle l_1 l_2 \rangle \langle l_2 1 \rangle \prod_{\alpha=1}^{n-1} \langle \alpha \alpha + 1 \rangle} \\
 &= A^{(0)}(\phi; 1^-, 2^-, \dots, n^+) \frac{\langle l_1 l_2 \rangle \langle n 1 \rangle}{\langle l_2 1 \rangle \langle n l_1 \rangle}.
 \end{aligned} \tag{3.7}$$

Applying a Schouten identity to the numerator and using momentum conservation in the form  $l_1 = l_2 + \widehat{P}_{1,n}$  we find,

$$A_L A_R = A^{(0)} \left( -\frac{N(\widehat{P}_{1,n}, p_1, p_n)}{(l_1 - p_n)^2 (l_2 + p_1)^2} - \frac{\widehat{P}_{1,n} \cdot p_n}{(l_1 - p_n)^2} + \frac{\widehat{P}_{1,n} \cdot p_1}{(l_2 + p_1)^2} \right), \tag{3.8}$$

where  $N(P, p_1, p_2) = P^2(p_1 \cdot p_2) - 2(P \cdot p_1)(P \cdot p_2)$ . This is now written in terms of scalar integrals so we can directly use the results of van Neerven [44] to perform the phase space integration:

$$\int d^D \text{LIPS}(-l_1, l_2, P) \frac{N(P, p_1, p_2)}{(l_1 + p_1)^2 (l_2 + p_2)^2} = \frac{c_\Gamma}{(4\pi)^2 \epsilon^2} 2i \sin(\pi\epsilon) \mu^{2\epsilon} |P^2|^{-\epsilon} \times {}_2F_1 \left( 1, -\epsilon; 1 - \epsilon; \frac{p_1 \cdot p_2 P^2}{N(P, p_1, p_2)} \right) \tag{3.9}$$

$$\int d^D \text{LIPS}(-l_1, l_2, P) \frac{2(P \cdot p_1)}{(l_1 + p_1)^2} = \frac{c_\Gamma}{(4\pi)^2 \epsilon^2} 2i \sin(\pi\epsilon) \mu^{2\epsilon} |P^2|^{-\epsilon} \tag{3.10}$$

$$\int d^D \text{LIPS}(-l_1, l_2, P) = -\frac{c_\Gamma}{(4\pi)^2 \epsilon (1 - 2\epsilon)} 2i \sin(\pi\epsilon) \mu^{2\epsilon} |P^2|^{-\epsilon} \tag{3.11}$$

where the factor  $c_\Gamma$  is given by,

$$c_\Gamma = (4\pi)^{\epsilon-2} \frac{\Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)}. \tag{3.12}$$

The final integration is over the  $z$  variable. However, the only dependence on  $z$  appears through the quantity  $\widehat{P}_{1,n}^3$  so it is convenient to make a change of variables,

$$\frac{dz}{z} = \frac{d(\widehat{P})^2}{P^2 - \widehat{P}^2} \tag{3.13}$$

to produce a dispersion integral that will re-construct the parts of the cut-constructible amplitude proportional to  $(s_{1,n})^{-\epsilon}$ ,

$$\int \frac{d(\widehat{P})^2}{P^2 - \widehat{P}^2} 2i \sin(\pi\epsilon) |\widehat{P}^2|^{-\epsilon} = 2\pi i (-P^2)^\epsilon. \tag{3.14}$$

The final result for this diagram then reads:

$$\mathcal{D}^{1,n} = \frac{c_\Gamma}{\epsilon^2} A^{(0)} \left( \frac{\mu^2}{-s_{1,n}} \right)^\epsilon \left( {}_2F_1 \left( 1, -\epsilon; 1 - \epsilon; \frac{p_1 \cdot p_n s_{1,n}}{N(P_{1,n}, p_1, p_n)} \right) + 1 \right). \tag{3.15}$$

---

<sup>3</sup>Through a suitable choice of  $\eta$ , one can always ensure that  $N(P, p_1, p_2)$  is independent of  $z$  [37]

The other “gluon-only” channels (figures 2(b) and (c)) reduce to scalar integrals in the same way and we merely quote the results,

$$\begin{aligned}
 \mathcal{D}^{1,n-1} = & \frac{c_\Gamma}{\epsilon^2} A^{(0)} \left( \frac{\mu^2}{-s_{1,n-1}} \right)^\epsilon \left( {}_2F_1 \left( 1, -\epsilon; 1 - \epsilon; \frac{p_1 \cdot p_n s_{1,n-1}}{N(P_{1,n-1}, p_1, p_n)} \right) \right. \\
 & + {}_2F_1 \left( 1, -\epsilon; 1 - \epsilon; \frac{p_n \cdot p_{n-1} s_{1,n-1}}{N(P_{1,n-1}, p_n, p_{n-1})} \right) \\
 & \left. - {}_2F_1 \left( 1, -\epsilon; 1 - \epsilon; \frac{p_1 \cdot p_{n-1} s_{1,n-1}}{N(P_{1,n-1}, p_1, p_{n-1})} \right) + 1 \right) \quad (3.16)
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{D}^{1,i} = & \frac{c_\Gamma}{\epsilon^2} A^{(0)} \left( \frac{\mu^2}{-s_{1,i}} \right)^\epsilon \left( {}_2F_1 \left( 1, -\epsilon; 1 - \epsilon; \frac{p_1 \cdot p_{i+1} s_{1,i}}{N(P_{1,i}, p_1, p_{i+1})} \right) \right. \\
 & + {}_2F_1 \left( 1, -\epsilon; 1 - \epsilon; \frac{p_n \cdot p_i s_{1,i}}{N(P_{1,i}, p_n, p_i)} \right) \\
 & - {}_2F_1 \left( 1, -\epsilon; 1 - \epsilon; \frac{p_1 \cdot p_i s_{1,i}}{N(P_{1,i}, p_1, p_i)} \right) \\
 & \left. - {}_2F_1 \left( 1, -\epsilon; 1 - \epsilon; \frac{p_n \cdot p_{i+1} s_{1,i}}{N(P_{1,i}, p_n, p_{i+1})} \right) \right). \quad (3.17)
 \end{aligned}$$

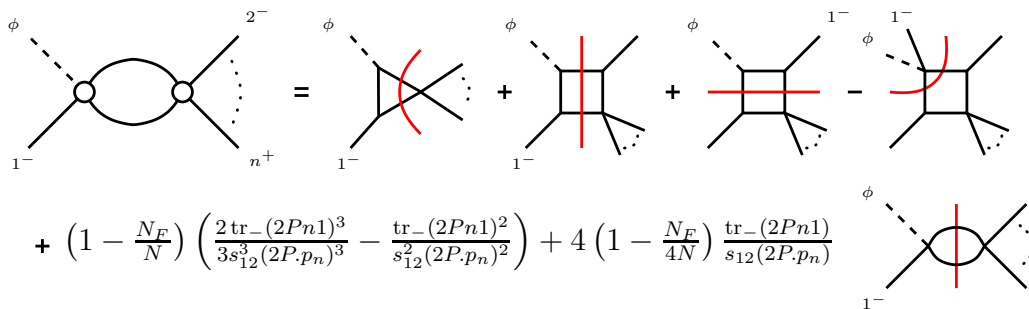
The arguments of these expressions can be straightforwardly permuted to produce results for channels 2(a),(b) and (c) for all gluon configurations.

When considering the channels with alternating helicity configurations around the loop we find that even after the Schouten identities have been applied, we are still left with tensor integrals which must be further reduced to scalar integrals by expanding in terms of all possible tensor structures. This feature has also been seen in the context of finding the cut-constructible part of pure QCD amplitudes and was also addressed applying Passarino-Veltman reduction [42]. For diagram 2(d), the  $s_{2,n}$  channel, the presence of tensor integrals and fermion loops results in new structures of order  $1/\epsilon$ . The result for this diagram is,

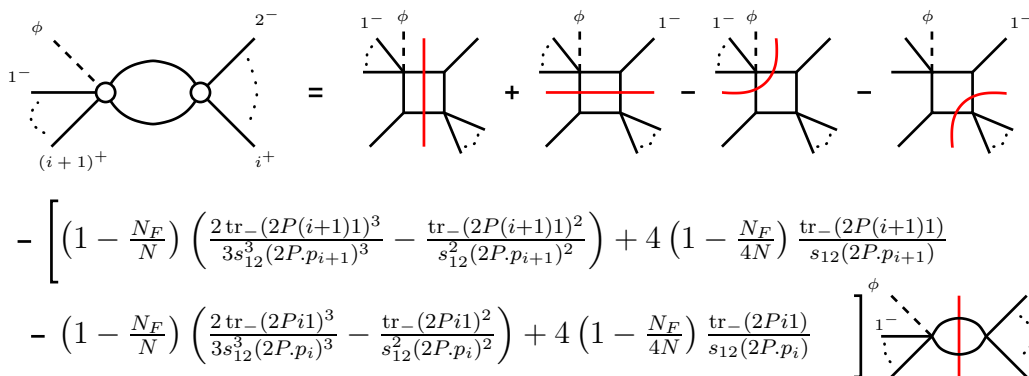
$$\begin{aligned}
 \mathcal{D}^{2,n} = & \frac{c_\Gamma}{\epsilon^2} A^{(0)} \left( \frac{\mu^2}{-s_{2,n}} \right)^\epsilon \left[ 1 + {}_2F_1 \left( 1, -\epsilon; 1 - \epsilon; \frac{p_2 \cdot p_n s_{2,n}}{N(P_{2,n}, p_2, p_n)} \right) \right. \\
 & + {}_2F_1 \left( 1, -\epsilon; 1 - \epsilon; \frac{p_1 \cdot p_n s_{2,n}}{N(P_{2,n}, p_1, p_n)} \right) \\
 & - {}_2F_1 \left( 1, -\epsilon; 1 - \epsilon; \frac{p_1 \cdot p_2 s_{2,n}}{N(P_{2,n}, p_1, p_2)} \right) \\
 & - \left( 1 - \frac{N_F}{N} \right) \left( \frac{2 \operatorname{tr}_-(2P_{3,n-1} n 1)^3}{3s_{12}^3 (2P \cdot n)^3} + \frac{\operatorname{tr}_-(P_{3,n-1} n 1)^2}{s_{12}^2 (2P \cdot n)^2} \right) \frac{\epsilon}{1 - 2\epsilon} \\
 & \left. - 4 \left( 1 - \frac{N_F}{4N} \right) \left( \frac{\operatorname{tr}_-(P_{3,n-1} n 1)}{s_{12} (2P \cdot n)} \right) \frac{\epsilon}{1 - 2\epsilon} \right], \quad (3.18)
 \end{aligned}$$

where we have introduced the shorthand notation,

$$\operatorname{tr}_-(abcd) = \langle a b \rangle [b c] \langle c d \rangle [d a]. \quad (3.19)$$



**Figure 3:** Decomposition of the MHV diagram of figure 2(d) contributing to the  $P_{2,n}$  channel



**Figure 4:** Decomposition of the MHV diagram of figure 2(e) contributing to the  $P_{2,i}$  channel

This is better illustrated in figure 3 which shows the cuts of each integral function that appear. Figure 4 shows the decomposition of the  $s_{2,i}$  channels (figure 2(e)) which follows exactly the same steps as the previous case. Diagrams 2(f) and 2(g) are analogous to diagrams 2(d) and 2(e) and can be found by permuting the arguments:  $1, 2, \dots, n \rightarrow 2, 1, n, \dots, 3$ .

When summing over all the possible diagrams we find that the bubble integrals always appear in the combination,

$$\text{Bub}(s) - \text{Bub}(t) = \mathcal{O}(\epsilon^0). \tag{3.20}$$

Hence all of the  $1/\epsilon$  poles coming from bubble functions vanish leaving the expected combination of boxes and triangles proportional to the tree amplitude [75, 76]. The combination of bubble integrals can be written in terms of a basis of pure logarithms,

$$L_k(s, t) = \frac{\log(s/t)}{(s-t)^k}. \tag{3.21}$$

These logarithmic contributions are not proportional to the tree amplitude, but are multiplied by new spinor structures written in terms of traces. The full, unrenormalised result

for this specific MHV helicity configuration for  $n \geq 3$  is thus:

$$\begin{aligned}
 C_n(\phi, 1^-, 2^-, 3^+ \dots, n^+) &= c_\Gamma A_n^{(0)}(\phi, 1^-, 2^-, 3^+, \dots, n^+) \left[ \sum_{i=1}^n (\mathbb{F}_3^{1m}(s_{i,n+i-2}) - \mathbb{F}_3^{1m}(s_{i,n+i-1})) \right. \\
 &\quad - \frac{1}{2} \sum_{i=1}^n \sum_{j=i+3}^{n+i-1} \mathbb{F}_4^{2me}(s_{i,j}, s_{i+1,j-1}; s_{i,j+1}, s_{i+1,j}) - \frac{1}{2} \sum_{i=1}^n \mathbb{F}_4^{1m}(s_{i,i+2}; s_{i,i+1}, s_{i+1,i+2}) \\
 &\quad + \sum_{i=4}^n \left( \frac{N_P \text{tr}_-(1P_{i,n}(i-1)2)^3}{3 s_{12}^3} L_3(s_{i-1,1}, s_{i,1}) + \frac{N_P \text{tr}_-(2P_{3,i-1}i1)^3}{3 s_{12}^3} L_3(s_{2,i}, s_{2,i-1}) \right. \\
 &\quad - \frac{N_P \text{tr}_-(1P_{i,n}(i-1)2)^2}{2 s_{12}^2} L_2(s_{i-1,1}, s_{i,1}) - \frac{N_P \text{tr}_-(2P_{3,i-1}i1)^2}{2 s_{12}^2} L_2(s_{2,i}, s_{2,i-1}) \\
 &\quad + \frac{N_P \text{tr}_-(1P_{i,n}(i-1)2)}{6 s_{12}} L_1(s_{i-1,1}, s_{i,1}) + \frac{N_P \text{tr}_-(2P_{3,i-1}i1)}{6 s_{12}} L_1(s_{2,i}, s_{2,i-1}) \\
 &\quad \left. + \frac{\beta_0 \text{tr}_-(1P_{i,n}(i-1)2)}{N s_{12}} L_1(s_{i-1,1}, s_{i,1}) + \frac{\beta_0 \text{tr}_-(2P_{3,i-1}i1)}{N s_{12}} L_1(s_{2,i}, s_{2,i-1}) \right) \Big], \tag{3.22}
 \end{aligned}$$

where the one-mass triangle  $\mathbb{F}_3^{1m}$  and box functions  $\mathbb{F}_4$  are defined in appendix B. For convenience, we have introduced

$$\beta_0 = \frac{11N - 2N_F}{3}, \quad N_P = 2 \left( 1 - \frac{N_F}{N} \right). \tag{3.23}$$

Note also that summations of the form  $\sum_a^b$  are understood to vanish when  $b < a$ . If we add this term together with its complex conjugate with the appropriate momentum re-labeling then we find agreement with the known Higgs MHV amplitude in the case of  $n = 3$  [77].

Eq. (3.22) can be rewritten in a form which is more convenient when computing the completion and overlap terms, namely

$$\begin{aligned}
 C_n(\phi, 1^-, 2^-, 3^+ \dots, n^+) &= c_\Gamma A_n^{(0)}(\phi, 1^-, 2^-, 3^+, \dots, n^+) \left[ \sum_{i=1}^n (\mathbb{F}_3^{1m}(s_{i,n+i-2}) - \mathbb{F}_3^{1m}(s_{i,n+i-1})) \right. \\
 &\quad \left. - \frac{1}{2} \sum_{i=1}^n \sum_{j=i+3}^{n+i-1} \mathbb{F}_4^{2me}(s_{i,j}, s_{i+1,j-1}; s_{i,j+1}, s_{i+1,j}) - \frac{1}{2} \sum_{i=1}^n \mathbb{F}_4^{1m}(s_{i,i+2}; s_{i,i+1}, s_{i+1,i+2}) \right] \\
 &\quad + \frac{c_\Gamma}{\Pi_{\alpha=2}^n \langle \alpha\alpha+1 \rangle} \sum_{i=4}^n \left[ \frac{N_P}{6} \langle 1P_{i,n}(i-1)2 \rangle \langle 1(i-1)P_{i,2}2 \rangle \left( \langle 1(i-1)P_{i,2}2 \rangle - \langle 1P_{i,n}(i-1)2 \rangle \right) \right. \\
 &\quad \times L_3(s_{i-1,1}, s_{i,1}) + \frac{\beta_0}{N} \langle 12 \rangle^2 \langle 1P_{i,n}(i-1)2 \rangle L_1(s_{i-1,1}, s_{i,1}) \\
 &\quad + \frac{N_P}{6} \langle 1iP_{3,i-1}2 \rangle \langle 1P_{1,i-1}i2 \rangle \left( \langle 1P_{1,i-1}i2 \rangle - \langle 1iP_{3,i-1}2 \rangle \right) \\
 &\quad \left. \times L_3(s_{2,i}, s_{2,i-1}) + \frac{\beta_0}{N} \langle 12 \rangle^2 \langle 1iP_{3,i-1}2 \rangle L_1(s_{2,i}, s_{2,i-1}) \right]. \tag{3.24}
 \end{aligned}$$

### 3.2 Cut-completion terms

The functions  $L_k(s, t)$  that appear in eqs. (3.22) and (3.24) are bubble contributions produced by the reduction of tensor box and triangle integrals. They contain unphysical

singularities as  $s \rightarrow t$  so it is useful to redefine the cut-containing contribution in terms of a new basis which has good behaviour in the various limits. This is achieved at the cost of adding some rational terms,

$$L_1(s, t) = \widehat{L}_1(s, t) \quad (3.25)$$

$$L_2(s, t) = \widehat{L}_2(s, t) + \frac{1}{2(s-t)} \left( \frac{1}{t} + \frac{1}{s} \right) \quad (3.26)$$

$$L_3(s, t) = \widehat{L}_3(s, t) + \frac{1}{2(s-t)^2} \left( \frac{1}{t} + \frac{1}{s} \right). \quad (3.27)$$

The new  $\widehat{L}_k$  functions are now free from spurious singularities. Replacing the  $L_k$  functions in (3.24) with the corresponding  $\widehat{L}_k$  functions we are left with the “completed cut term”:

$$\begin{aligned} CR_n(\phi, 1^-, 2^-, 3^+, \dots, n^+) &= \frac{c_\Gamma N_P}{12 \Pi_{\alpha=2}^n (\alpha\alpha + 1)} \sum_{i=4}^n \left[ \langle 1P_{i,n}(i-1)2 \rangle \langle 1(i-1)P_{i,2}2 \rangle \right. \\ &\quad \times \left( \langle 1(i-1)P_{i,2}2 \rangle - \langle 1P_{i,n}(i-1)2 \rangle \right) \\ &\quad \times \frac{1}{(s_{i-1,1} - s_{i,1})^2} \left( \frac{1}{s_{i-1,1}} + \frac{1}{s_{i,1}} \right) \\ &\quad + \langle 1iP_{3,i-1}2 \rangle \langle 1P_{1,i-1}i2 \rangle \left( \langle 1P_{1,i-1}i2 \rangle - \langle 1iP_{3,i-1}2 \rangle \right) \\ &\quad \left. \times \frac{1}{(s_{2,i} - s_{2,i-1})^2} \left( \frac{1}{s_{2,i}} + \frac{1}{s_{2,i-1}} \right) \right]. \quad (3.28) \end{aligned}$$

#### 4. The rational contributions

By definition, the rational terms only contain poles in the invariants and therefore are amenable to the same type of analysis as used for tree-amplitudes. We will therefore calculate the remaining rational terms in  $A_n^{(1)}(\phi, 1^-, 2^-, \dots, n^+)$  using on-shell recursion relations that have been successful in both QCD [49–51, 57–59] and for the finite amplitudes involving the  $\phi$  field [62].

This approach relies on both the factorisation properties of one-loop amplitudes on physical poles and the introduction of a complex shift parameter  $z$  to study the behaviour of the amplitude in the complex plane.

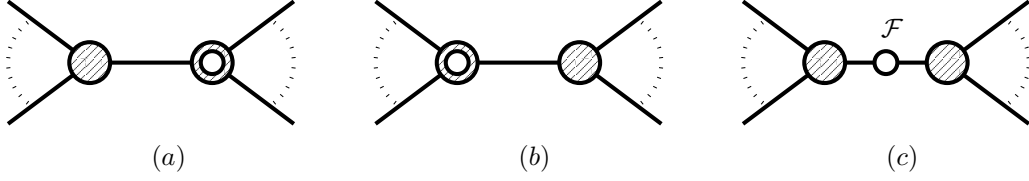
We recall the multi-particle factorisation properties of one-loop amplitudes [78],

$$A_n^{(1)} \rightarrow \frac{A_L^{(0)} A_R^{(1)}}{P_i^2} + \frac{A_L^{(1)} A_R^{(0)}}{P_i^2} + \mathcal{F} \frac{A_L^{(0)} A_R^{(0)}}{P_i^2} \quad \text{as } P_i^2 \rightarrow 0, \quad (4.1)$$

where the subscripts  $L$  and  $R$  denote the amplitudes with fewer external particles on the left and right of the factorising physical channel as shown in figure 5. Note that  $\mathcal{F}$  only contributes in multi-particle channels if the tree amplitude contains a pole in that channel. MHV amplitudes do not have multi-particle poles and hence this term is absent.

Following the discussion in the earlier sections, we divide the amplitude into its cut-constructible and rational terms

$$A_n^{(1)} = C_n + R_n. \quad (4.2)$$



**Figure 5:** Factorisation of one-loop amplitudes in the limit  $P_i^2 \rightarrow 0$ . Contribution (c) does not appear for MHV amplitudes.

Applying eq. (4.1) in the rational sector we find (for MHV amplitudes),

$$R_n \rightarrow \frac{A_L^{(0)} R_R}{P_i^2} + \frac{R_L A_R^{(0)}}{P_i^2} \quad \text{as } P_i^2 \rightarrow 0. \quad (4.3)$$

However, as it stands,  $R_n$  contains unphysical poles which cancel with unphysical poles in  $C_n$ . Therefore, we add the rational function  $CR_n$  (the cut completion term of section 3.2) to  $C_n$  and subtract it from  $R_n$ .  $CR_n$  is chosen such that it cancels the unphysical poles in  $C_n$  (and thus, simultaneously in  $R_n$ ). This means that we should try to set up a recursion on the physical poles with the expression  $\widehat{R}_n = R_n - CR_n$ .

To develop the recursion, a convenient analytic continuation is to shift the spinors of the negative helicity gluons 1 and 2 such that

$$|\widehat{1}\rangle = |1\rangle + z|2\rangle, \quad |\widehat{2}\rangle = |2\rangle - z|1\rangle. \quad (4.4)$$

The corresponding momenta are also shifted,

$$p_1^\mu \rightarrow p_1^\mu(z) = p_1^\mu + \frac{z}{2} \langle 2 | \gamma^\mu | 1 \rangle, \quad p_2^\mu \rightarrow p_2^\mu(z) = p_2^\mu - \frac{z}{2} \langle 2 | \gamma^\mu | 1 \rangle. \quad (4.5)$$

We now consider the integral,

$$\frac{1}{2\pi i} \oint_C \frac{dz}{z} \widehat{R}_n(z) = \frac{1}{2\pi i} \oint_C \frac{dz}{z} (R_n(z) - CR_n(z)). \quad (4.6)$$

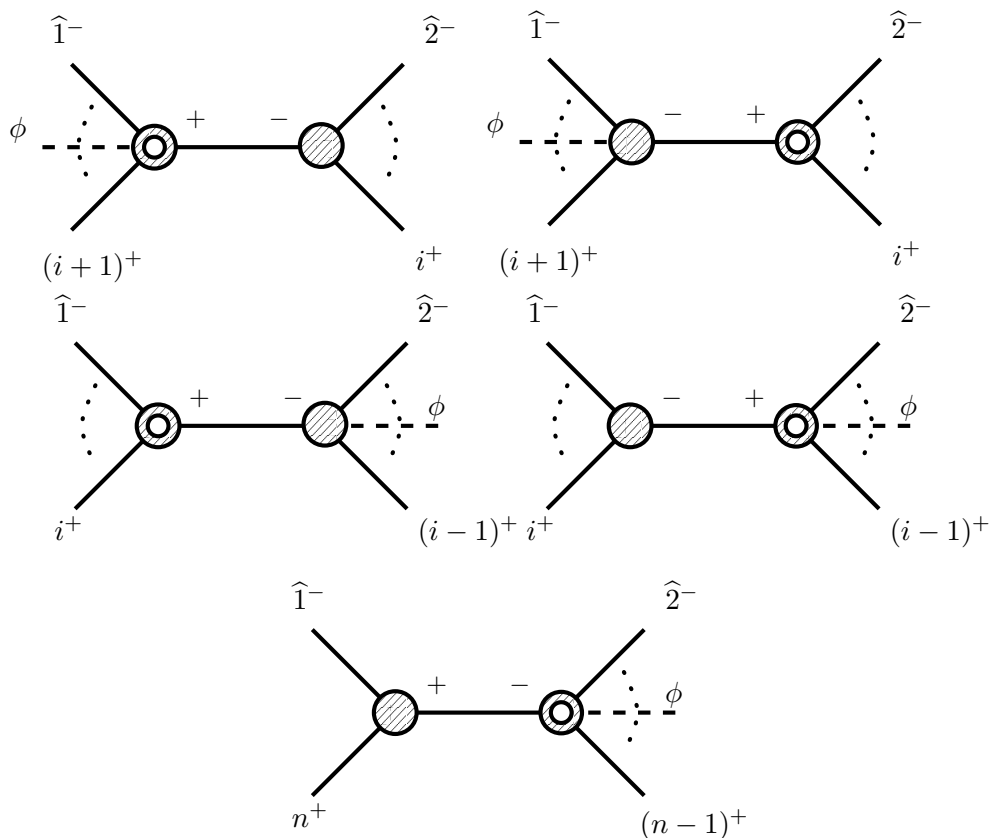
Assuming that there is no surface term at infinity, the integral vanishes. The remaining residues are fixed by the multiparticle factorisation (4.1) so that the rational contribution is given by

$$\begin{aligned} \widehat{R}_n(0) &= - \sum_{\text{phys. poles } z_i} \text{Res}_{z=z_i} \frac{(R_n(z) - CR_n(z))}{z} \\ &= \sum_i \frac{A_L^{(0)}(z) R_R(z) + R_L(z) A_R^{(0)}(z)}{P_i^2} + \sum_i \text{Res}_{z=z_i} \frac{CR_n(z)}{z} \end{aligned} \quad (4.7)$$

The last term is called the overlap term. It is quite simple to calculate if the poles at physical  $z_i$  are all first order, as this makes them similar to recursive terms.

To sum up, the rational terms consist of the recursive terms which are similar in calculation to tree-level, completion terms which can be computed simply from the cut-containing functions, and the overlap terms which can be computed by considering the completion terms as certain internal momenta go on-shell.





**Figure 6:** The direct recursive diagrams contributing to  $R_n(\phi, 1^-, 2^-, \dots, n^+)$  with a  $|1|\langle 2|$  shift.

### 4.1 Recursive terms

The recursive part of the rational contribution is defined by,

$$R_n^D = \sum_i \frac{A_L^{(0)}(z)R_R(z) + R_L(z)A_R^{(0)}(z)}{P_i^2}. \tag{4.8}$$

For the choice of shift given in eq. (4.4) the allowed types of contributing diagrams contributions are shown in figure 6.

Combining the various diagrams, we find that recursive terms obey the following rela-

tion,

$$\begin{aligned}
 R_n^D(\phi; 1^-, 2^-, 3^+, \dots, n^+) &= \sum_{i=4}^{n-1} R(\phi; \widehat{1}^-, \widehat{P}_{2,i}^+, (i+1)^+, \dots, n^+) \frac{1}{s_{2,i}} A^{(0)}(-\widehat{P}_{2,i}^-, \widehat{2}^-, \dots, i^+) \\
 &+ \sum_{i=4}^n A^{(0)}(\phi; \widehat{1}^-, \widehat{P}_{2,i}^-, (i+1)^+, \dots, n^+) \frac{1}{s_{2,i}} R(-\widehat{P}_{2,i}^+, \widehat{2}^-, \dots, i^+) \\
 &+ \sum_{i=3}^{n-1} R(\widehat{1}^-, -\widehat{P}_{i,1}^+, i^+, \dots, n^+) \frac{1}{s_{i,1}} A^{(0)}(\phi; \widehat{P}_{i,1}^-, \widehat{2}^-, \dots, (i-1)^+) \\
 &+ \sum_{i=4}^{n-1} A^{(0)}(\widehat{1}^-, -\widehat{P}_{i,1}^-, i^+, \dots, n^+) \frac{1}{s_{i,1}} R(\phi; \widehat{P}_{i,1}^+, \widehat{2}^-, \dots, (i-1)^+) \\
 &+ R(\phi; \widehat{1}^-, \widehat{P}_{23}^+, 4^+, \dots, n^+) \frac{1}{s_{23}} A^{(0)}(-\widehat{P}_{23}^-, \widehat{2}^-, 3^+) \\
 &+ A^{(0)}(\widehat{1}^-, -\widehat{P}_{n1}^+, n^+) \frac{1}{s_{n1}} R(\phi; \widehat{P}_{n1}^-, \widehat{2}^- \dots, (n-1)^+). \quad (4.9)
 \end{aligned}$$

Here  $R$  represents the full rational part of the one-loop amplitude with fewer external legs. The contributions with  $i = n$  in the first term and  $i = 3$  in the fourth term are absent because  $A_2(\phi, -, +)$  (and hence  $R_2(\phi, -, +)$ ) is zero by conservation of angular momentum.

Note that one could have written down other recursive terms that contain three-point pure gauge amplitudes. The most problematic of these are those where the “one-loop”-ness is in the gauge three-point, since the factorisation properties for complex momenta are not fully understood. However, when the two external gluons have opposite helicity it is guaranteed to vanish, because the corresponding splitting function does not have rational parts. For our particular choice of shift, the only possible pairings of the external gluons are  $(2^-, 3^+)$  or  $(n^+, 1^-)$ , and we find this significant simplification in all possible cases.

This leaves the cases where pure gauge three-point is at tree level. Because of the way we have chosen our shift, the amplitudes

$$A^0(n^+, \widehat{1}^-, -\widehat{P}_{n,1}^-), \quad A^0(\widehat{2}^-, 3^+, -\widehat{P}_{2,3}^+) \quad (4.10)$$

both vanish, so that the only two contributions involving a three gluon vertex are when  $i = 3$  in the first term and the last term of eq. (4.9). This latter contribution is the “homogenous” term in the recursion; it depends on the  $\phi$ -MHV amplitude with one gluon fewer. The first two  $\phi$ -MHV amplitudes are known,

$$R_2(\phi; 1^-, 2^-) = \frac{1}{8\pi^2} A^{(0)}(\phi, 1^-, 2^-), \quad (4.11)$$

$$R_3(\phi; 1^-, 2^-, 3^+) = \frac{1}{8\pi^2} A^{(0)}(\phi, 1^-, 2^-, 3^+). \quad (4.12)$$

Because the tree amplitudes with fewer than two negative helicities vanish, the remaining one-loop contributions needed are those with one negative helicity. These are finite one-loop amplitudes and are entirely rational. The finite  $\phi - + \dots +$  amplitudes were computed for arbitrary numbers of positive helicity gluons in ref. [62]. As a concrete

example, the three-gluon amplitude is given by,

$$R_3(\phi; 1^-, 2^+, 3^+) = \frac{N_P}{96\pi^2} \frac{\langle 12 \rangle \langle 31 \rangle [23]}{\langle 23 \rangle^2} - \frac{1}{8\pi^2} A_3^{(0)}(\phi^\dagger; 1^-, 2^+, 3^+). \quad (4.13)$$

Similarly, the pure QCD  $- + \cdots +$  amplitudes are given to all orders in ref. [79, 49]. In the four gluon case, the result is,

$$R_4(1^-, 2^+, 3^+, 4^+) = \frac{N_P}{96\pi^2} \frac{\langle 24 \rangle [24]^3}{[12] \langle 23 \rangle \langle 34 \rangle [41]} \quad (4.14)$$

The value that  $z$  takes is obtained by requiring that the shifted momenta

$$\widehat{P}_{i,j}^\mu = P_{i,j}^\mu \pm \frac{z}{2} \langle 2 | \gamma^\mu | 1 \rangle, \quad (4.15)$$

is on-shell. In this equation, the sign is positive when the momentum set  $\{p_i, p_j\}$  includes  $p_1$  and is negative when it includes  $p_2$ . There are four distinct channels, specified by the multiple invariants  $s_{i,1}$ ,  $s_{2,i}$  and the double invariants  $s_{n1}$  and  $s_{23}$ . In each channel, we find that the value of  $z$  and the hatted variables are given by,

$$\begin{aligned} s_{i,1} \text{ channels:} \quad z_{i,1} &= -\frac{s_{i,1}}{\langle 2 | P_{i,2} | 1 \rangle}, \\ \widehat{1} &= -\frac{|P_{i,n} P_{i,2} 2 \rangle}{[1 P_{i,1} 2]}, \quad \widehat{2} = \frac{|P_{i,2} P_{i,n} 1 \rangle}{\langle 2 P_{i,1} 1 \rangle}, \quad \widehat{P}_{i,1} = \frac{|P_{i,n} 1 \rangle \langle 2 P_{i,2} |}{\langle 2 P_{i,1} 1 \rangle}, \end{aligned} \quad (4.16)$$

$$\begin{aligned} s_{2,i} \text{ channels:} \quad z_{2,i} &= \frac{s_{2,i}}{\langle 2 | P_{3,i} | 1 \rangle}, \\ \widehat{1} &= \frac{|P_{1,i} P_{3,i} 2 \rangle}{[1 P_{2,i} 2]}, \quad \widehat{2} = -\frac{|P_{3,i} P_{1,i} 1 \rangle}{\langle 2 P_{2,i} 1 \rangle}, \quad \widehat{P}_{2,i} = \frac{|P_{1,i} 1 \rangle \langle 2 P_{3,i} |}{\langle 2 P_{2,i} 1 \rangle}, \end{aligned} \quad (4.17)$$

$$\begin{aligned} s_{n1} \text{ channel:} \quad z_{n1} &= -\frac{\langle 1n \rangle}{\langle 2n \rangle}, \\ \widehat{1} &= |n \rangle \frac{\langle 1 2 \rangle}{\langle n 2 \rangle}, \quad \widehat{2} = \frac{|P_{1,2} n \rangle}{\langle 2 n \rangle}, \quad \widehat{P}_{n,1} = \frac{|n \rangle \langle 2 P_{n,1} |}{\langle 2 n \rangle}, \end{aligned} \quad (4.18)$$

$$\begin{aligned} s_{23} \text{ channel:} \quad z_{23} &= \frac{[32]}{[31]}, \\ \widehat{1} &= \frac{|P_{1,2} 3 \rangle}{[1 3]}, \quad \widehat{2} = |3 \rangle \frac{[12]}{[1 3]}, \quad \widehat{P}_{2,3} = \frac{|P_{2,3} 1 \rangle [3]}{[31]}. \end{aligned} \quad (4.19)$$

## 4.2 Overlap terms

The overlap terms are defined by [58]

$$O_n = \sum_i \text{Res}_{z=z_i} \frac{CR_n(z)}{z}. \quad (4.20)$$

They can be obtained by evaluating the residue of the cut completion term  $CR_n$  given in eq. (3.28) in each of the physical channels. In order to make the residue calculation straightforward, it is convenient to use identities such as  $s_{i,j} - s_{i,j-1} = \langle j | P_{i,j-1} | j \rangle$  to

rewrite eq. (3.28) in a way that exposes each of the physical poles,

$$\begin{aligned}
 CR_n(\phi, 1^-, 2^-, 3^+, \dots, n^+) &= \frac{c_{\Gamma NP}}{12 \langle n1 \rangle \Pi_{\alpha=2}^{n-1} \langle \alpha\alpha + 1 \rangle} \\
 &\left[ \frac{\langle 1|43|2 \rangle \langle 1|P_{2,3}4|2 \rangle \left( \langle 1|P_{2,3}4|2 \rangle - \langle 14 \rangle [43] \langle 32 \rangle \right)}{s_{23} \langle 4|P_{2,3}|4 \rangle^2} \right. \\
 &+ \frac{\langle 1|nP_{2,n}|2 \rangle \langle 1|P_{2,n}n|2 \rangle \left( \langle 1|P_{2,n}n|2 \rangle - \langle 1nP_{2,n}|2 \rangle \right)}{s_{2,n} \langle n|P_{2,n}|n \rangle^2} \\
 &+ \frac{\langle 1|P_{3,1}3|2 \rangle \langle 1|3P_{3,1}|2 \rangle \left( \langle 1|3P_{3,1}|2 \rangle - \langle 1P_{3,1}3|2 \rangle \right)}{s_{3,1} \langle 3|P_{3,1}|3 \rangle^2} \\
 &+ \frac{\langle 1|n(n-1)|2 \rangle \langle 1|(n-1)P_{n,1}|2 \rangle \left( \langle 1|(n-1)P_{n,1}|2 \rangle - \langle 1|n(n-1)|2 \rangle \right)}{s_{n1} \langle (n-1)|P_{n,1}|(n-1) \rangle^2} \\
 &+ \sum_{i=4}^{n-1} \frac{1}{s_{i,1}} \left( \frac{\langle 1|P_{i,1}i|2 \rangle \langle 1|iP_{i,1}|2 \rangle \left( \langle 1|iP_{i,1}|2 \rangle - \langle 1P_{i,1}i|2 \rangle \right)}{\langle i|P_{i,1}|i \rangle^2} \right. \\
 &\quad \left. + \frac{\langle 1|P_{i,1}(i-1)|2 \rangle \langle 1|(i-1)P_{i,1}|2 \rangle \left( \langle 1|(i-1)P_{i,1}|2 \rangle - \langle 1P_{i,1}(i-1)|2 \rangle \right)}{\langle (i-1)|P_{i,1}|(i-1) \rangle^2} \right) \\
 &+ \sum_{i=4}^{n-1} \frac{1}{s_{2,i}} \left( \frac{\langle 1|iP_{2,i}|2 \rangle \langle 1|P_{2,i}i|2 \rangle \left( \langle 1|P_{2,i}i|2 \rangle - \langle 1iP_{2,i}|2 \rangle \right)}{s_{2,i} \langle i|P_{2,i}|i \rangle^2} \right. \\
 &\quad \left. + \frac{\langle 1|(i+1)P_{2,i}|2 \rangle \langle 1|P_{2,i}(i+1)|2 \rangle \left( \langle 1|P_{2,i}(i+1)|2 \rangle - \langle 1|(i+1)P_{2,i}|2 \rangle \right)}{s_{2,i} \langle (i+1)|P_{2,i}|(i+1) \rangle^2} \right) \left. \right]. \tag{4.21}
 \end{aligned}$$

We observe that the cut completion term contains only simple residues so for the  $P_{i,j}$  pole, the overlap term is given by,

$$O_n^{i,j} = CR_n(z_{i,j}) \frac{\widehat{s}_{i,j}}{s_{i,j}} \tag{4.22}$$

where  $z_{i,j}$  is the value of  $z$  that puts  $\widehat{P}_{i,j}$  on-shell. The multiplicative factor removes the  $\widehat{s}_{i,j}$  pole in  $CR_n$  and replaces it with the correct propagator  $s_{i,j}$ . Note that the only terms that are affected by the momentum shifts are  $|1\rangle$ ,  $|2\rangle$  and any invariant including either  $p_1$  or  $p_2$ . The overall factor  $\langle n1 \rangle$  must be treated carefully, but not  $\langle 23 \rangle$ .

Let us first consider the  $s_{23}$  pole. The overlap term is given by,

$$\begin{aligned}
 O_n^{23} &= \frac{CR(z_{23}) \widehat{s}_{23}}{s_{23}} \\
 &= \frac{c_{\Gamma NP}}{12} \frac{\langle \widehat{14} \rangle [43] \langle \widehat{1} | \widehat{P}_{2,3} 4 | 2 \rangle \left( \langle \widehat{1} | \widehat{P}_{2,3} 4 | 2 \rangle - \langle \widehat{14} \rangle [43] \langle 32 \rangle \right)}{[23] \langle n\widehat{1} \rangle \langle 4 | \widehat{P}_{2,3} | 4 \rangle^2 \Pi_{\alpha=2}^{n-1} \langle \alpha\alpha + 1 \rangle}. \tag{4.23}
 \end{aligned}$$

Employing the definitions of the shifted variables in the  $s_{23}$  channel given in eq. (4.19), we find that

$$O_n^{23} = -\frac{N_P}{192\pi^2} \frac{s_{123}[34]\langle 42\rangle\langle 4|P_{12}|3\rangle(s_{123}\langle 42\rangle + \langle 4|P_{12}|3\rangle\langle 32\rangle)}{s_{23}\langle n|P_{12}|3\rangle\langle 4|P_{23}|1\rangle^2 \Pi_{\alpha=3}^{n-1}\langle \alpha\alpha + 1\rangle}. \quad (4.24)$$

Similarly, we find that

$$O_n^{2n} = -\frac{N_P}{192\pi^2} \frac{s_{1,n}\langle 2n\rangle\langle 2|P_{3,n}|n\rangle(\langle 2|P_{3,n}P_{1,n}|n\rangle + s_{1,n}\langle 2n\rangle)}{s_{2,n}\langle n|P_{1,n}|1\rangle^2 \Pi_{\alpha=2}^{n-1}\langle \alpha\alpha + 1\rangle}, \quad (4.25)$$

$$O_n^{31} = \frac{N_P}{192\pi^2} \frac{s_{3,n}\langle 32\rangle\langle 2|P_{3,1}|3\rangle\langle 2|P_{3,1}P_{3,n}|3\rangle(\langle 2|P_{3,1}P_{3,n}|3\rangle + s_{3,n}\langle 23\rangle)}{s_{3,1}\langle n|P_{3,n}P_{3,1}|2\rangle\langle 3|P_{3,1}|1\rangle^2 \Pi_{\alpha=2}^{n-1}\langle \alpha\alpha + 1\rangle}. \quad (4.26)$$

The overlap in the  $s_{n1}$  channel is complicated by the fact that there is an overall factor of  $1/\langle n1\rangle$ . We find,

$$\begin{aligned} O_n^{n1} = & \frac{N_P}{192\pi^2} \frac{\langle 12\rangle^3}{\langle n1\rangle \Pi_{\alpha=2}^{n-1}\langle \alpha\alpha + 1\rangle} \left[ \frac{[n(n-1)]\langle (n-1)2\rangle}{[n1]\langle 12\rangle} + \right. \\ & \sum_{i=4}^{n-1} \left\{ \frac{\langle n|P_{i,n}(i-1)|2\rangle\langle n|(i-1)P_{i,2}|2\rangle(\langle n|(i-1)P_{i,2}|2\rangle - \langle n|P_{i,n}(i-1)|2\rangle)}{\langle n|(i-1)P_{i,2} + P_{i,n}(i-1)|2\rangle^2} \right. \\ & \quad \times \left( \frac{1}{\langle n|P_{i-1,n}P_{i-1,2}|2\rangle} + \frac{1}{\langle n|P_{i,n}P_{i,2}|2\rangle} \right) \\ & \quad + \frac{\langle n|iP_{3,i-1}|2\rangle\langle n|P_{1,i-1}i|2\rangle(\langle n|P_{1,i-1}i|2\rangle - \langle n|iP_{3,i-1}|2\rangle)}{\langle n|iP_{3,i-1} + P_{1,i-1}i|2\rangle^2} \\ & \quad \left. \left. \times \left( \frac{1}{\langle n|P_{1,i}P_{3,i}|2\rangle} + \frac{1}{\langle n|P_{1,i-1}P_{3,i-1}|2\rangle} \right) \right\} \right]. \quad (4.27) \end{aligned}$$

Finally, when  $i \leq n-1$ , the overlap terms are given by

$$\begin{aligned} O_n^{i,1} = & \frac{N_P}{192\pi^2} \frac{s_{i,n}}{s_{i,1}\langle n|P_{i,n}P_{i,2}|2\rangle \Pi_{\alpha=2}^{n-1}\langle \alpha\alpha + 1\rangle} \left[ \right. \\ & \frac{\langle 2|P_{i,2}(i-1)|2\rangle\langle 2|P_{i,2}P_{i,n}|(i-1)\rangle(\langle 2|P_{i,2}P_{i,n}|(i-1)\rangle + s_{i,n}\langle 2(i-1)\rangle)}{\langle (i-1)|P_{i,n}|1\rangle^2} \\ & \left. + \frac{\langle 2|P_{i,2}i|2\rangle\langle 2|P_{i,2}P_{i,n}|i\rangle(\langle 2|P_{i,2}P_{i,n}|i\rangle + s_{i,n}\langle 2i\rangle)}{\langle i|P_{i,n}|1\rangle^2} \right], \quad (4.28) \end{aligned}$$

$$\begin{aligned} O_n^{2,i} = & \frac{N_P}{192\pi^2} \frac{s_{1,i}}{s_{2,i}\langle n|P_{1,i}P_{3,i}|2\rangle \Pi_{\alpha=2}^{n-1}\langle \alpha\alpha + 1\rangle} \left[ \right. \\ & \frac{\langle 2i\rangle\langle 2|P_{3,i}|i\rangle\langle 2|P_{3,i}P_{1,i}|i\rangle(\langle 2|P_{3,i}P_{1,i}|i\rangle + s_{1,i}\langle 2i\rangle)}{\langle i|P_{1,i}|1\rangle^2} \\ & \left. + \frac{\langle 2(i+1)\rangle\langle 2|P_{3,i}|(i+1)\rangle\langle 2|P_{3,i}P_{1,i}|(i+1)\rangle(\langle 2|P_{3,i}P_{1,i}|(i+1)\rangle + s_{1,i}\langle 2(i+1)\rangle)}{\langle (i+1)|P_{1,i}|1\rangle^2} \right]. \quad (4.29) \end{aligned}$$

## 5. The 4-point amplitude

We will now compute the rational contribution explicitly for  $A^{(1)}(\phi : 1^-, 2^-, 3^+, 4^+)$ . The recursion relation (4.9) consists of four physical channels corresponding to poles in the invariants  $s_{23}$ ,  $s_{234}$ ,  $s_{41}$  and  $s_{341}$ .

Using the spinor shifts given in eq. (4.19) together with the known amplitudes given in eqs. (4.11)–(4.14) it is straightforward to evaluate the direct recursive terms,

$$R_4(\phi; 1^-, 2^-, 3^+, 4^+) = R_4^{234} + R_4^{23} + R_4^{341} + R_4^{41}. \quad (5.1)$$

For example, in the  $s_{234}$  channel, we find,

$$\begin{aligned} R_4^{234} &= A^0(\phi, \widehat{1}^-, \widehat{P}_{234}^-) \times \frac{1}{s_{234}} \times R(\widehat{2}^-, 3^+, 4^+, -\widehat{P}_{234}^+) \\ &= \frac{N_{PC\Gamma}}{6} \frac{\langle \widehat{1} \widehat{P}_{234} \rangle^2 \langle 3 \widehat{P}_{234} \rangle [3 \widehat{P}_{234}]^3}{s_{234} [\widehat{2} 3] \langle 3 4 \rangle \langle 4 \widehat{P}_{234} \rangle [\widehat{P}_{234} \widehat{2}]}. \end{aligned} \quad (5.2)$$

Employing the definitions of the shifts, (4.19), we easily find

$$R_4^{234} = -\frac{N_P}{96\pi^2} \frac{[43] s_{1234}^2 \langle 24 \rangle^3 \langle 3P_{24}1 \rangle}{\langle 34 \rangle^2 s_{234} \langle 2P_{34}1 \rangle \langle 4P_{23}1 \rangle^2}. \quad (5.3)$$

Similarly,

$$R_4^{23} = -\frac{N_P}{96\pi^2} \frac{s_{123} [34] [31] \langle 4|P_{12}|3 \rangle}{[23] [12] \langle 4|P_{23}|1 \rangle^2} - \frac{1}{8\pi^2} A^{(0)}(\phi^\dagger, 1^-, 2^-, 3^+, 4^+), \quad (5.4)$$

$$R_4^{41} = \frac{1}{8\pi^2} A^{(0)}(\phi, 1^-, 2^-, 3^+, 4^+), \quad (5.5)$$

$$R_4^{341} = \frac{N_P}{96\pi^2} \frac{[31] \langle 2|P_{13}|4 \rangle^3}{s_{341} \langle 34 \rangle [41]^2 \langle 2|P_{34}|1 \rangle}. \quad (5.6)$$

Notice that the expressions for  $R_4^{23}$  and  $R_4^{41}$  are not proportional to  $N_P$ . It is clear that these two terms will cancel against similar terms appearing in the  $\phi^\dagger$ -MHV amplitude so that the Higgs amplitude does indeed have this property.

The overlap terms are defined by,

$$O_4(\phi; 1^-, 2^-, 3^+, 4^+) = O_4^{234} + O_4^{23} + O_4^{341} + O_4^{41}. \quad (5.7)$$

The individual contributions can be obtained by setting  $n = 4$  in eqs. (4.24), (4.26), (4.26) and (4.27) respectively. After some trivial simplifications, we find,

$$O_4^{23} = -\frac{N_P}{192\pi^2} \frac{s_{123} [34] \langle 42 \rangle (s_{123} \langle 42 \rangle + \langle 4|P_{12}|3 \rangle \langle 32 \rangle)}{s_{23} \langle 34 \rangle \langle 4|P_{23}|1 \rangle^2}, \quad (5.8)$$

$$O_4^{234} = -\frac{N_P}{192\pi^2} \frac{[34] \langle 42 \rangle m_H^2 (m_H^2 \langle 42 \rangle - \langle 2|P_{34}P_{123}|4 \rangle)}{s_{234} \langle 34 \rangle \langle 4P_{23}1 \rangle^2}, \quad (5.9)$$

$$O_4^{341} = -\frac{N_P}{192\pi^2} \frac{[34] \langle 2|P_{13}|4 \rangle (\langle 2|P_{13}|4 \rangle + \langle 23 \rangle [34])}{s_{341} \langle 34 \rangle [41]^2}, \quad (5.10)$$

$$O_4^{41} = -\frac{N_P}{192\pi^2} \frac{[34] \langle 12 \rangle^2}{s_{41} \langle 34 \rangle}. \quad (5.11)$$

With a little further algebra it is possible to write the sum of recursive and overlap terms in a form which is free of spurious singularities,

$$\begin{aligned} \widehat{R}_4(\phi, 1^-, 2^-, 3^+, 4^+) &= R_4(\phi; 1^-, 2^-, 3^+, 4^+) + O_4(\phi; 1^-, 2^-, 3^+, 4^+) \tag{5.12} \\ &= \frac{1}{8\pi^2} A^{(0)}(A; 1^-, 2^-, 3^+, 4^+) + \frac{N_P[43]}{96\pi^2 \langle 34 \rangle} \left[ \right. \\ &\quad - \frac{\langle 23 \rangle \langle 1|P_{24}|3 \rangle^2}{\langle 34 \rangle [43] [32] s_{234}} + \frac{\langle 41 \rangle \langle 3|P_{12}|3 \rangle}{\langle 34 \rangle [12] [32]} - \frac{\langle 14 \rangle \langle 2|P_{13}|4 \rangle^2}{\langle 34 \rangle [43] [41] s_{341}} + \frac{\langle 32 \rangle \langle 4|P_{12}|4 \rangle}{\langle 34 \rangle [12] [41]} \\ &\quad \left. + \frac{\langle 12 \rangle^2}{\langle 34 \rangle [43]} - \frac{\langle 12 \rangle}{[12]} - \frac{\langle 12 \rangle \langle 2|P_{13}|4 \rangle}{2[41] s_{341}} + \frac{\langle 12 \rangle \langle 1|P_{24}|3 \rangle}{2[32] s_{234}} + \frac{\langle 12 \rangle^2}{2s_{23}} + \frac{\langle 12 \rangle^2}{2s_{41}} \right], \end{aligned}$$

where the pseudoscalar amplitude  $A^{(0)}(A; 1^-, 2^-, 3^+, 4^+)$  is given by the difference of  $\phi$  and  $\phi^\dagger$  amplitudes.

To form the full 1-loop  $\phi$ -MHV amplitude we need to add this to the cut-constructible piece given in eq. (3.24) and the completion term of eq. (4.21).

$$\begin{aligned} A_4^{(1)}(\phi, 1^-, 2^-, 3^+, 4^+) &= C_4(\phi, 1^-, 2^-, 3^+, 4^+) + CR_4(\phi, 1^-, 2^-, 3^+, 4^+) \\ &\quad + \widehat{R}_4(\phi, 1^-, 2^-, 3^+, 4^+). \tag{5.13} \end{aligned}$$

As discussed earlier, the Higgs amplitude is constructed from the sum of the  $\phi$  and the  $\phi^\dagger$  amplitudes where the  $\phi^\dagger$  contribution is obtained using parity symmetry,

$$\begin{aligned} A_4^{(1)}(H, 1^-, 2^-, 3^+, 4^+) &= A_4^{(1)}(\phi, 1^-, 2^-, 3^+, 4^+) + A_4^{(1)}(\phi^\dagger, 1^-, 2^-, 3^+, 4^+) \\ &= A_4^{(1)}(\phi, 1^-, 2^-, 3^+, 4^+) + \left( A_4^{(1)}(\phi^\dagger, 3^-, 4^-, 1^+, 2^+) \right)_{\langle ij \rangle \leftrightarrow [ji]} \tag{5.14} \end{aligned}$$

The one-loop amplitudes in this paper are computed in the four-dimensional helicity scheme and are not renormalised. To perform an  $\overline{MS}$  renormalisation, one should subtract an  $\overline{MS}$  counterterm from  $A_n^{(1)}$ ,

$$A_n^{(1)} \rightarrow A_n^{(1)} - c_\Gamma \frac{n \beta_0}{2 \epsilon} A_n^{(0)}. \tag{5.15}$$

The Wilson coefficient (2.2) produces an additional finite contribution,

$$A_n^{(1)} \rightarrow A_n^{(1)} + \frac{11}{2} A_n^{(0)}. \tag{5.16}$$

## 6. Cross checks and limits

### 6.1 Infra-red pole structure

The infra-red poles are constrained to have a certain form proportional to the tree level amplitude [75, 76],

$$A_n^{(1)} = -\frac{c_\Gamma}{\epsilon^2} A_n^{(0)} \sum_{i=1}^n \left( \frac{\mu^2}{-s_{ii+1}} \right)^\epsilon + \mathcal{O}(\epsilon^0). \tag{6.1}$$

Expanding the hypergeometric functions as a series in  $\epsilon$  quickly leads to a proof of this fact, and it can be seen that all logarithms vanish at order  $1/\epsilon$ .

## 6.2 Collinear limits

In general the collinear behaviour of one-loop amplitudes can be written [19, 80]:

$$\begin{aligned}
 A_n^{(1)}(\dots, i^{\lambda_i}, i+1^{\lambda_{i+1}}, \dots) &\xrightarrow{i||i+1} \\
 \sum_{h=\pm} A_{n-1}^{(1)}(\dots, i-1^{\lambda_{i-1}}, P^h, i+2^{\lambda_{i+2}}, \dots) &\text{Split}^{(0)}(-P^{-h}; i^{\lambda_i}, i+1^{\lambda_{i+1}}) \\
 + A_{n-1}^{(0)}(\dots, i-1^{\lambda_{i-1}}, P^h, i+2^{\lambda_{i+2}}, \dots) &\text{Split}^{(1)}(-P^{-h}; i^{\lambda_i}, i+1^{\lambda_{i+1}})
 \end{aligned} \tag{6.2}$$

where the collinear limit is defined through  $p_i \rightarrow zP$  and  $p_{i+1} \rightarrow (1-z)P$ . The universal splitting functions for QCD have been calculated in reference [19, 20, 81]. The tree-level splitting functions are given by,

$$\text{Split}^{(0)}(-P^+, 1^-, 2^+) = \frac{z^2}{\sqrt{z(1-z)}\langle 12 \rangle} \tag{6.3}$$

$$\text{Split}^{(0)}(-P^+, 1^+, 2^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}\langle 12 \rangle} \tag{6.4}$$

$$\text{Split}^{(0)}(-P^-, 1^+, 2^+) = \frac{1}{\sqrt{z(1-z)}\langle 12 \rangle} \tag{6.5}$$

$$\text{Split}^{(0)}(-P^-, 1^-, 2^-) = 0. \tag{6.6}$$

It is convenient to divide the one-loop splitting functions into cut-constructible and rational components,

$$\text{Split}^{(1)}(-P^{-h}, 1^{\lambda_1}, 2^{\lambda_2}) = \text{Split}^{(1),C}(-P^{-h}, 1^{\lambda_1}, 2^{\lambda_2}) + \text{Split}^{(1),R}(-P^{-h}, 1^{\lambda_1}, 2^{\lambda_2}) \tag{6.7}$$

where

$$\begin{aligned}
 \text{Split}^{(1),C}(-P^\pm, 1^-, 2^+) &= \text{Split}^{(0)}(-P^\pm, 1^-, 2^+) \frac{\epsilon_\Gamma}{\epsilon^2} \times \\
 &\left( \frac{\mu^2}{-s_{12}} \right)^\epsilon \left( 1 - {}_2F_1 \left( 1, -\epsilon; 1-\epsilon; \frac{z}{z-1} \right) - {}_2F_1 \left( 1, -\epsilon; 1-\epsilon; \frac{z-1}{z} \right) \right),
 \end{aligned} \tag{6.8}$$

$$\begin{aligned}
 \text{Split}^{(1),C}(-P^+, 1^-, 2^-) &= \text{Split}^{(0)}(-P^+, 1^-, 2^-) \frac{\epsilon_\Gamma}{\epsilon^2} \times \\
 &\left( \frac{\mu^2}{-s_{12}} \right)^\epsilon \left( 1 - {}_2F_1 \left( 1, -\epsilon; 1-\epsilon; \frac{z}{z-1} \right) - {}_2F_1 \left( 1, -\epsilon; 1-\epsilon; \frac{z-1}{z} \right) \right),
 \end{aligned} \tag{6.9}$$

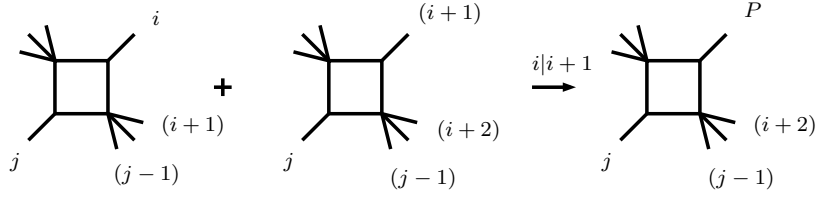
$$\text{Split}^{(1),C}(-P^-, 1^-, 2^-) = 0, \tag{6.10}$$

$$\text{Split}^{(1),R}(-P^\pm, 1^-, 2^+) = 0, \tag{6.11}$$

$$\text{Split}^{(1),R}(-P^+, 1^-, 2^-) = \frac{N_P}{96\pi^2} \frac{\sqrt{z(1-z)}}{[12]}, \tag{6.12}$$

$$\text{Split}^{(1),R}(-P^-, 1^-, 2^-) = \frac{N_P}{96\pi^2} \frac{\sqrt{z(1-z)}\langle 12 \rangle}{[12]^2}. \tag{6.13}$$





**Figure 7:** Factorisation of combinations of two-mass “easy” box functions in limit where  $i \parallel i + 1$ .

### 6.3 Collinear factorisation of the cut-constructible contributions

Considering only the cut constructible contributions, we expect the following limiting behaviour

$$\begin{aligned}
 & C_n^{(1)}(\dots, i^{\lambda_i}, i + 1^{\lambda_{i+1}}, \dots) \xrightarrow{i \parallel i+1} \\
 & \sum_{h=\pm} C_{n-1}^{(1)}(\dots, i - 1^{\lambda_{i-1}}, P^h, i + 2^{\lambda_{i+2}}, \dots) \text{Split}^{(0)}(-P^{-h}; i^{\lambda_i}, i + 1^{\lambda_{i+1}}) \\
 & + A_{n-1}^{(0)}(\dots, i - 1^{\lambda_{i-1}}, P^h, i + 2^{\lambda_{i+2}}, \dots) \text{Split}^{(1,C)}(-P^{-h}; i^{\lambda_i}, i + 1^{\lambda_{i+1}}) \quad (6.14)
 \end{aligned}$$

There are three collinear limits to consider for the  $\phi$ -MHV amplitude; two collinear negative helicity gluons, two collinear positive helicity gluons and the mixed case with one of each helicity.

We first note that the box and triangle scalar integrals always appear in the combination

$$\begin{aligned}
 U_n = & \left[ \sum_{i=1}^n (F_3^{1m}(s_{i,n+i-2}) - F_3^{1m}(s_{i,n+i-1})) - \frac{1}{2} \sum_{i=1}^n \sum_{j=i+3}^{n+i-1} F_4^{2me}(s_{i,j}, s_{i+1,j-1}; s_{i,j+1}, s_{i+1,j}) \right. \\
 & \left. - \frac{1}{2} \sum_{i=1}^n F_4^{1m}(s_{i,i+2}; s_{i,i+1}, s_{i+1,i+2}) \right] \quad (6.15)
 \end{aligned}$$

and multiply the tree amplitude.

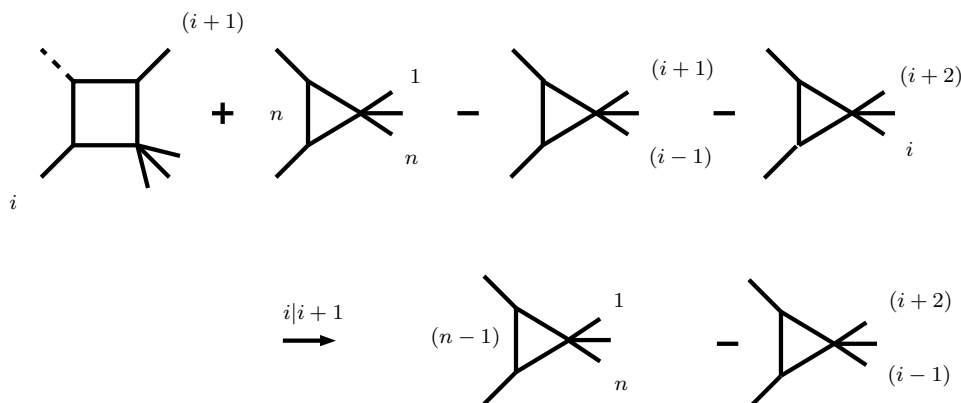
Many of the collinear factorisation properties of the individual box and triangle functions have been studied in ref. [19] for example,

$$\begin{aligned}
 & F_4^{2me}(s_{i,j}, s_{i+1,j-1}; s_{i,j-1}, s_{i+1,j}) + F_4^{2me}(s_{i+1,j}, s_{i+2,j-1}; s_{i+1,j-1}, s_{i+2,j}) \\
 & \xrightarrow{i \parallel i+1} F_4^{2me}(s_{P,j}, s_{i+2,j-1}; s_{i+2,j}, s_{P,j-1}), \quad (6.16)
 \end{aligned}$$

where  $s_{P,j} = (P + p_{i+2} + \dots + p_j)^2$  and which is illustrated in figure 7. A similar relation applies when  $j = i + 3$  such that the second and third terms are one-mass boxes.

In much the same way, the particular combination of box and one-mass triangle functions shown in figure 8 factorise onto triangle functions as follows,

$$\begin{aligned}
 & F_4^{2me}(s_{1,n}, s_{i+2,i-1}; s_{i+1,i-1}, s_{i+2,i}) + n F_3^{1m}(s_{1,n}) - F_3^{1m}(s_{i+1,i-1}) - F_3^{1m}(s_{i+2,i}) \\
 & \xrightarrow{i \parallel i+1} (n - 1) F_3^{1m}(s_{1,n}) - F_3^{1m}(s_{i+2,i-1}). \quad (6.17)
 \end{aligned}$$



**Figure 8:** Factorisation of a combination of two-mass “easy” box and one-mass triangle functions in limit where  $i \parallel i + 1$ . Note that the scale relevant for the triangle functions is indicated on the right of the triangle. The massless legs on the left of the triangles do not correspond to physical momenta.

Taken together, it is straightforward to find the collinear limit of  $U_n$

$$\begin{aligned}
 U_n(1, \dots, i, i + 1, \dots, n) &\xrightarrow{i \parallel i + 1} U_{n-1}(1, \dots, P, \dots, n) \\
 &+ \frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s_{i,i+1}} \right)^\epsilon \left( 1 - {}_2F_1 \left( 1, -\epsilon; 1 - \epsilon; \frac{z}{z-1} \right) - {}_2F_1 \left( 1, -\epsilon; 1 - \epsilon; \frac{z-1}{z} \right) \right),
 \end{aligned} \tag{6.18}$$

which is independent of which (adjacent) pair are taken collinear.

The remaining limits depend on the helicity of the collinear gluons. To explore the limits in detail, we find it convenient to utilise the expression for  $C_n$  given in eq. (3.22).

### 6.3.1 The mixed helicity collinear limit

Let us now consider the case where we take a negative and a positive helicity collinear, e.g. the  $2 \parallel 3$  limit shown in figure 9. We immediately see that the third diagram involves a factorisation onto a finite  $\phi$  amplitude which has no cut-constructible part and therefore gives no contribution.

The finite logarithms also factorise in the expected way since,

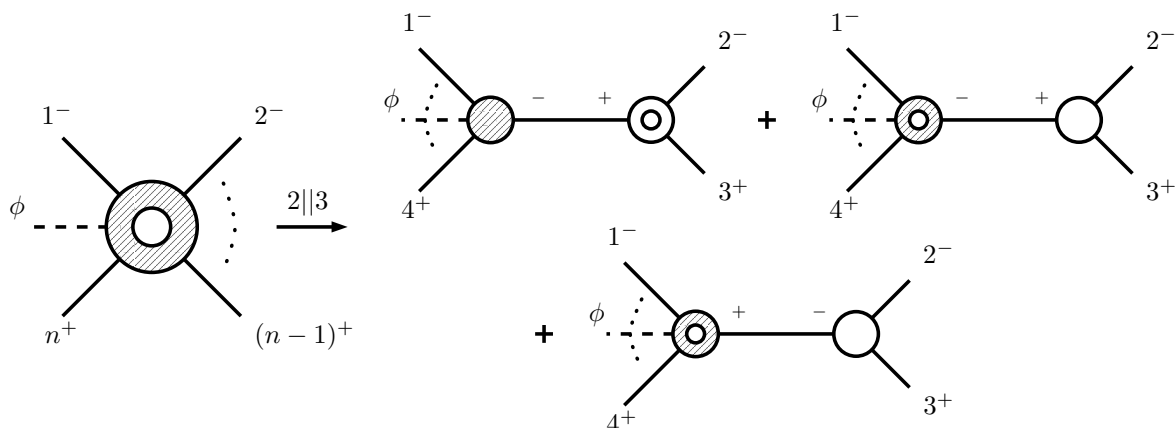
$$\frac{\text{tr}_-(1P_{i,n}(i-1)2)}{s_{12}} \xrightarrow{2 \parallel 3} \frac{\text{tr}_-(1P_{i,n}(i-1)P)}{s_{1P}} \quad i > 4, \tag{6.19}$$

$$\frac{\text{tr}_-(2P_{3,i-1}i1)}{s_{12}} \xrightarrow{2 \parallel 3} \frac{\text{tr}_-(PP_{4,i-1}i1)}{s_{1P}} \quad i > 4, \tag{6.20}$$

while

$$\frac{\text{tr}_-(1P_{4,n}32)}{s_{12}} \xrightarrow{2 \parallel 3} 0 \tag{6.21}$$

$$\frac{\text{tr}_-(2341)}{s_{12}} \xrightarrow{2 \parallel 3} 0 \tag{6.22}$$



**Figure 9:** Collinear factorisation of  $A^{(1)}(\phi; 1^-, 2^-, 3^+, \dots, n^+)$  taking  $p_2$  and  $p_3$  parallel.

ensures that terms with divergent logarithms, i.e.  $L_k(s_{234}, s_{23})$ , will always be proportional to a trace which vanishes in the limit and hence do not appear in the one-loop splitting function. Together with eq. (6.18), we find that, as expected,

$$C_n(\phi; 1^-, 2^-, 3^+, \dots, n^+) \xrightarrow{2||3} A_{n-1}^{(0)}(\phi; 1^-, P^-, 4^+, \dots, n^+) \text{Split}^{(1),C}(-P^+, 2^-, 3^+) + C_{n-1}(\phi; 1^-, P^-, 4^+, \dots, n^+) \text{Split}^{(0)}(-P^+, 2^-, 3^+). \quad (6.23)$$

### 6.3.2 Two collinear negative helicity gluons

The  $1||2$  limit is rather trivial since there is only a single, finite term in this case as shown in figure 10. The tree amplitude in this case vanishes,

$$A_n^{(0)}(1^-, 2^-, 3^+, \dots, n^+) \xrightarrow{1||2} 0, \quad (6.24)$$

therefore it is obvious that all the box and triangle terms of eq. (3.24) will vanish. The remaining finite logs appear to have a singularity in  $s_{12}$ , the worst coming from the traces raised to the 3rd power,

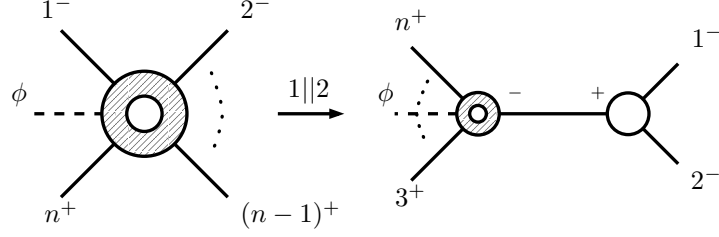
$$\frac{\text{tr}_-(1XY2)^3}{s_{12}^3} = \frac{\langle 1|XY|2 \rangle^3}{\langle 12 \rangle^3}, \quad \frac{\text{tr}_-(2XY1)^3}{s_{12}^3} = -\frac{\langle 2|XY|1 \rangle^3}{\langle 12 \rangle^3}. \quad (6.25)$$

However the tree amplitude is proportional to  $\langle 12 \rangle^3$  so  $C_n(\phi, 1^-, 2^-, 3^+, \dots, n^+)$  vanishes in the limit as expected.

### 6.3.3 Two collinear positive helicity gluons

The final limit occurs when we take any adjacent pair of positive helicities collinear. The relevant diagrams are shown in figure 11. We can drop the third contribution since it involves the purely rational splitting function.

We must also be able to show that there are no divergent terms coming from the finite logs and that these terms correctly factorise onto the lower point amplitude. This turns out



**Figure 10:** Collinear factorisation of  $A^{(1)}(\phi; 1^-, 2^-, 3^+, \dots, n^+)$  taking  $p_1$  and  $p_2$  parallel. The only contribution is finite so the cut-constructible parts vanish in this limit.

to be slightly more involved than in the  $2||3$  case. First let us choose to take two adjacent particles  $a$  and  $b$  collinear where  $p_a$  lies to the left of  $p_b$  in the clockwise ordering. Using,

$$s_{b,i} \xrightarrow{a||b} (1-z)s_{P,i} + zs_{b+1,i}, \quad (6.26)$$

$$s_{i,a} \xrightarrow{a||b} zs_{i,P} + (1-z)s_{i,a-1}, \quad (6.27)$$

it is then possible to show:

$$\begin{aligned} \frac{\text{tr}_-(2P_{3,a-1}a1)^k}{s_{12}^k} L_k(s_{2,a}, s_{2,a-1}) + \frac{\text{tr}_-(2P_{3,a}b1)^k}{s_{12}^k} L_k(s_{2,b}, s_{2,a}) \\ \xrightarrow{a||b} \frac{\text{tr}_-(2P_{3,a-1}P1)^k}{s_{12}^k} L_k(s_{2,P}, s_{2,a-1}), \end{aligned} \quad (6.28)$$

and,

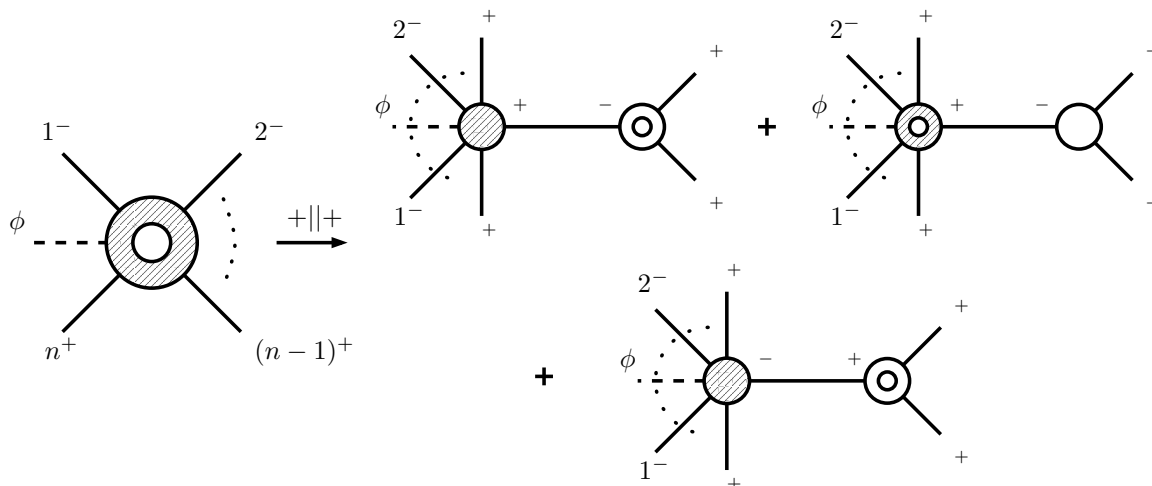
$$\begin{aligned} \frac{\text{tr}_-(1P_{b+1,i}b2)^k}{s_{12}^k} L_k(s_{b,i}, s_{b+1,i}) + \frac{\text{tr}_-(1P_{b,i}a2)^k}{s_{12}^k} L_k(s_{a,i}, s_{b,i}) \\ \xrightarrow{a||b} \frac{\text{tr}_-(1P_{b+1,i}P2)^k}{s_{12}^k} L_k(s_{P,i}, s_{b+1,i}). \end{aligned} \quad (6.29)$$

Note that  $s_{2,P} = (p_2 + \dots + p_{a-1} + P)^2$  and  $s_{P,i} = (P + p_{b+1} + \dots + p_i)^2$ . Using these identities and recognising that  $\text{tr}_-(1ab2) \xrightarrow{a||b} 0$  and together with eq. (6.18), it is straightforward to show that eq. (3.24) has the correct factorisation properties,

$$\begin{aligned} C_n(\phi; 1^-, 2^-, 3^+, \dots, a^+, b^+, \dots, n^+) \xrightarrow{a||b} \\ C_{n-1}(\phi; 1^-, 2^-, 3^+, \dots, a-1^+, P^+, b+1^+, \dots, n^+) \text{Split}^{(0)}(-P^-, a^+, b^+) \\ + \mathcal{A}_{n-1}^{(0)}(\phi; 1^-, 2^-, 3^+, \dots, a-1^+, P^+, b+1^+, \dots, n^+) \text{Split}^{(1,C)}(-P^-, a^+, b^+). \end{aligned} \quad (6.30)$$

### 6.3.4 Collinear factorisation of the rational contributions

In this section, we focus on the collinear limit of the rational part of the 4 gluon amplitude with two negative and two positive helicities. There are three independent collinear limits,  $1||2$ ,  $2||3$  and  $3||4$ .



**Figure 11:** Collinear factorisation of  $A^{(1)}(\phi; 1^-, 2^-, 3^+, \dots, n^+)$  taking any two positive helicities parallel.

Let us first consider the case where  $1^-$  and  $2^-$  are parallel. From figure 10 we see that there is only a single contributing term and it is fairly straightforward to show that eq. (5.13) satisfies,

$$R_4(\phi; 1^-, 2^-, 3^+, 4^+) \xrightarrow{1||2} \frac{1}{8\pi^2} R_3(\phi^\dagger; P^-, 3^+, 4^+) \text{Split}^{(0)}(-P^+, 1^-, 2^-). \quad (6.31)$$

The  $2^-||3^+$  limit, shown in figure 9, has only two contributing diagrams for the raitonal piece since the first contribution vanishes since the one-loop mixed helicity splitting function has no rational part. The rational part of the four gluon amplitude can easily be shown to satisfy,

$$R_4(\phi; 1^-, 2^-, 3^+, 4^+) + CR_4(\phi; 1^-, 2^-, 3^+, 4^+) \xrightarrow{2||3} R_3(\phi^\dagger; 1^-, P^+, 4^+) \text{Split}^{(0)}(-P^-, 2^-, 3^+) + R_3(\phi; 1^-, P^-, 4^+) \text{Split}^{(0)}(-P^+, 2^-, 3^+) \quad (6.32)$$

The final collinear limit is given by taking  $3^+$  and  $4^+$  collinear. Here we again find the expected behaviour,

$$R_4(\phi; 1^-, 2^-, 3^+, 4^+) \xrightarrow{3||4} R_3(\phi; 1^-, 2^-, P^+) \text{Split}^{(0)}(-P^-, 3^+, 4^+) + A^{(0)}(\phi; 1^-, 2^-, P^+) \text{Split}^{(1),R}(-P^-, 3^+, 4^+) + A^{(0)}(\phi; 1^-, 2^-, P^-) \text{Split}^{(1),R}(-P^+, 3^+, 4^+). \quad (6.33)$$

### 6.4 Soft Higgs limit

For the case of a massless Higgs boson, we can consider the kinematic limit  $p_H \rightarrow 0$ . Because of the form of the  $HG_{\mu\nu}G^{\mu\nu}$  interaction, the Higgs field behaves like a constant in this limit, so the Higgs-plus- $n$ -gluon amplitudes should be related to pure gauge theory

amplitudes. Low energy theorems relate the amplitudes with zero Higgs momentum to pure gauge theory amplitudes [82]:

$$A_n^{(l)}(H, \{g_i, \lambda_i\}) \xrightarrow{p_H \rightarrow 0} Cg \frac{\partial}{\partial g} A_n^{(l)}(\{g_i, \lambda_i\}), \quad (6.34)$$

where  $C$  is the effective coupling of Higgs field to the gluon fields. The  $n$ -gluon tree amplitude is proportional to  $g^{n-2}$  (see equation (2.9)) therefore,

$$A_n^{(0)}(H, \{g_i, \lambda_i\}) \xrightarrow{p_H \rightarrow 0} (\text{const.}) \times (n-2) A_n^{(0)}(\{g_i, \lambda_i\}). \quad (6.35)$$

In terms of the  $\phi$  and  $\phi^\dagger$  components [32],

$$A_n^{(0)}(\phi, \{g_i, \lambda_i\}) \xrightarrow{p_\phi \rightarrow 0} (\text{const.}) \times (n_- - 1) A_n^{(0)}(\{g_i, \lambda_i\}), \quad (6.36)$$

$$A_n^{(0)}(\phi^\dagger, \{g_i, \lambda_i\}) \xrightarrow{p_{\phi^\dagger} \rightarrow 0} (\text{const.}) \times (n_+ - 1) A_n^{(0)}(\{g_i, \lambda_i\}) \quad (6.37)$$

where  $n_+$  and  $n_-$  are the number of positive and negative helicity particles respectively.

The one-loop amplitudes are proportional to  $g^n$  hence similarly one can deduce the following behaviour in the soft Higgs limit,

$$A_n^{(1)}(H, \{g_i, \lambda_i\}) \xrightarrow{p_H \rightarrow 0} (\text{const.}) \times n A_n^{(1)}(\{g_i, \lambda_i\}). \quad (6.38)$$

It has been conjectured in [62] that the  $\phi$  and  $\phi^\dagger$  components generalise from the tree level relations to give,

$$A_n^{(1)}(\phi, \{g_i, \lambda_i\}) \xrightarrow{p_\phi \rightarrow 0} (\text{const.}) \times n_- A_n^{(1)}(\{g_i, \lambda_i\}), \quad (6.39)$$

$$A_n^{(1)}(\phi^\dagger, \{g_i, \lambda_i\}) \xrightarrow{p_{\phi^\dagger} \rightarrow 0} (\text{const.}) \times n_+ A_n^{(1)}(\{g_i, \lambda_i\}). \quad (6.40)$$

The 4-gluon MHV amplitude at one-loop in QCD has been derived in ref. [83] and is given, unrenormalised in the four-dimensional helicity scheme by

$$C_4(1^-, 2^-, 3^+, 4^+) = -\frac{2c_\Gamma}{\epsilon^2} A^{(0)}(1^-, 2^-, 3^+, 4^+) \left[ \left( \frac{\mu^2}{-s_{14}} \right)^\epsilon + \left( \frac{\mu^2}{-s_{12}} \right)^\epsilon + \log^2 \left( \frac{s_{12}}{s_{14}} \right) + \pi^2 - \frac{\beta_0}{N\epsilon(1-2\epsilon)} \left( \frac{\mu^2}{-s_{14}} \right)^\epsilon \right], \quad (6.41)$$

$$R_4(1^-, 2^-, 3^+, 4^+) = \frac{N_{PC\Gamma}}{9} A^{(0)}(1^-, 2^-, 3^+, 4^+). \quad (6.42)$$

#### 6.4.1 Soft limit of $A_4^{(1)}(\phi, 1^-, 2^-, 3^+, 4^+)$

Firstly let us consider the soft limit of the cut constructible components, eq. (3.24). The 1-mass and 2-mass easy box functions and triangle functions have smooth soft limits where we take:

$$\left( \frac{\mu^2}{-m_\phi^2} \right)^\epsilon \xrightarrow{p_\phi \rightarrow 0} 0 \quad (6.43)$$

$$\left( \frac{\mu^2}{-s_{\phi i}} \right)^\epsilon \xrightarrow{p_\phi \rightarrow 0} 0. \quad (6.44)$$

We must apply the same relations to the finite logs coming from the tensor triangle integrals, for instance we find:

$$L_k(s_{341}, s_{41}) = \frac{\text{Bub}(s_{341}) - \text{Bub}(s_{41})}{(s_{341} - s_{41})^k} \xrightarrow{p_\phi \rightarrow 0} \frac{(-1)^k}{s_{14}^k \epsilon (1 - 2\epsilon)} \left( \frac{\mu^2}{-s_{14}} \right)^\epsilon. \quad (6.45)$$

Applying these relations together with momentum conservation to the cut-constructible part of the 4 gluon amplitude we find that the boxes and triangle functions collapse onto the correct  $1/\epsilon^2$  poles while the coefficients of the bubble contributions simplify considerably, e.g.,

$$\left( \frac{\text{tr}_-(1432)}{s_{12}} \right)^k \xrightarrow{p_\phi \rightarrow 0} (-1)^k s_{14}^k, \quad (6.46)$$

so that all of the logarithms proportional to  $N_P$  cancel among themselves leaving,

$$\left( \frac{\mu^2}{-s_{14}} \right)^\epsilon \frac{2\beta_0}{N\epsilon(1 - 2\epsilon)}. \quad (6.47)$$

Combining this with the  $1/\epsilon^2$  poles we find,

$$C_4(\phi; 1^-, 2^-, 3^+, 4^+) \xrightarrow{p_\phi \rightarrow 0} 2 C_4(1^-, 2^-, 3^+, 4^+) \quad (6.48)$$

The soft limit of the rational part, eq. (5.13) and eq. (3.28), gives

$$R_4(\phi; 1^-, 2^-, 3^+, 4^+) \xrightarrow{p_\phi \rightarrow 0} -\frac{N_P c_T}{3} A^{(0)}(1^-, 2^-, 3^+, 4^+), \quad (6.49)$$

with all the poles in the triple invariants vanishing. This implies that the rational terms vanish in the limit for the full Higgs field,

$$R_4(H; 1^-, 2^-, 3^+, 4^+) \xrightarrow{p_\phi \rightarrow 0} 0, \quad (6.50)$$

since the  $\phi^\dagger$  contribution appears with the opposite sign. This confirms that cut-constructible terms of the amplitude do follow the naive factorisation proposed in [62] but the rational parts do not. This seems to be consistent with the results for the  $Hq\bar{q}Q\bar{Q}$  amplitude presented in [64].

## 7. Conclusions

Recent developments based on unitarity and on-shell recursion relations have lead to important breakthroughs in computing non-supersymmetric one-loop gauge theory amplitudes. In particular, new compact analytic results have been obtained for gluonic amplitudes involving six or more gluons.

In this paper we have focused on amplitudes involving gluons and a colourless scalar — the Higgs boson. The model which we use to calculate these amplitudes is the tree-level pure gauge theory plus an effective interaction  $HG_{\mu\nu}G^{\mu\nu}$  produced by considering the heavy top quark limit of the one-loop coupling of Higgs and gluons in the non-supersymmetric standard model. Following ref. [32], we split the interaction into selfdual and anti-selfdual

pieces. The self-dual (anti-self-dual) gauge fields interact with the  $\phi$  ( $\phi^\dagger$ ) scalars respectively and because of this selfduality, the amplitudes for  $\phi$  plus  $n$  gluons, and those for  $\phi^\dagger$  plus  $n$  gluons, each have a simpler structure than the gluonic amplitudes for either  $H$  or  $A$ .

Previous studies using the unitarity-factorisation bootstrap program have focused on the  $\phi$  amplitudes that are finite at one-loop, i.e. the all positive or nearly all positive helicity [62] or the divergent amplitudes when the gluons all have negative helicity [63]. In this paper we have employed four-dimensional unitarity and recursion relations to compute the one-loop corrections to amplitudes involving a colourless scalar  $\phi$ , two colour adjacent negative helicity gluons and an arbitrary number of positive helicity gluons - the so-called  $\phi$ -MHV amplitudes.

The gluonic production of Higgs bosons via a heavy quark loop is expected to be the largest source of Higgs bosons at the LHC. Because of the size of the strong coupling, it will be important to understand Higgs plus jet events in some detail. The two jet channel has contributions of both QCD [72, 73, 84–89] and weak boson fusion origin [90–92]. Separating these two “signals” is of crucial importance for measuring the coupling of the Higgs bosons to standard model particles from LHC data [93]. The amplitudes presented here may be useful in computing the gluon fusion contamination of the weak boson fusion signal.

The one-loop amplitudes naturally divide into cut-containing  $C_n$  and rational parts  $R_n$ . We used the double cut unitarity approach of ref. [37] to derive all the multiplicity results for  $C_n$  given in eq. (3.24). The rational terms have several sources - first the cut-completion term  $CR_n$  which eliminates the unphysical poles present in  $C_n$ , second the direct on-shell recursion contribution  $R_n^D$  and third the overlap term  $O_n$ . Explicit formulae for these contributions are given in eqs. (3.28), (4.9) and (4.24)–(4.29). An explicit solution for the four gluon case is given in section 5. We have checked our results in the limit where two of the gluons are collinear, in the limit where the scalar becomes soft and against previously known results for up to four gluons.

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## A. Spinor conventions

In the spinor helicity formalism [94–99] an on-shell momentum of a massless particle,  $k_\mu k^\mu = 0$ , is represented as

$$k_{\alpha\dot{\alpha}} \equiv k_\mu \sigma_{\alpha\dot{\alpha}}^\mu = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}, \tag{A.1}$$



where  $\lambda_\alpha$  and  $\tilde{\lambda}_{\dot{\alpha}}$  are two commuting spinors of positive and negative chirality. Spinor inner products are defined by

$$\langle \lambda, \lambda' \rangle = \epsilon_{\alpha\beta} \lambda^\alpha \lambda'^\beta, \quad [\tilde{\lambda}, \tilde{\lambda}'] = -\epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}^{\dot{\alpha}} \tilde{\lambda}'^{\dot{\beta}}, \quad (\text{A.2})$$

and a scalar product of two null vectors,  $k_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$  and  $p_{\alpha\dot{\alpha}} = \lambda'_\alpha \tilde{\lambda}'_{\dot{\alpha}}$ , becomes

$$k_\mu p^\mu = -\frac{1}{2} \langle \lambda, \lambda' \rangle [\tilde{\lambda}, \tilde{\lambda}']. \quad (\text{A.3})$$

We use the shorthand  $\langle ij \rangle$  and  $[ij]$  for the inner products of the spinors corresponding to momenta  $p_i$  and  $p_j$ ,

$$\langle ij \rangle = \langle \lambda_i, \lambda_j \rangle, \quad [ij] = [\tilde{\lambda}_i, \tilde{\lambda}_j]. \quad (\text{A.4})$$

For gluon polarization vectors we use

$$\varepsilon_\mu^\pm(k, \xi) = \pm \frac{\langle \xi^\mp | \gamma_\mu | k^\mp \rangle}{\sqrt{2} \langle \xi^\mp | k^\pm \rangle}, \quad (\text{A.5})$$

where  $k$  is the gluon momentum and  $\xi$  is the reference momentum, an arbitrary null vector which can be represented as the product of two reference spinors,  $\xi_{\alpha\dot{\alpha}} = \xi_\alpha \tilde{\xi}_{\dot{\alpha}}$ .

## B. Scalar integrals

The one-loop functions that appear in the all-orders cut-constructible contribution  $C_n$  given in eq. (3.24) are defined by,

$$F_4^{0m}(s, t) = \frac{2}{\epsilon^2} \left[ \left( \frac{\mu^2}{-s} \right)^\epsilon {}_2F_1 \left( 1, -\epsilon; 1-\epsilon; -\frac{u}{t} \right) + \left( \frac{\mu^2}{-t} \right)^\epsilon {}_2F_1 \left( 1, -\epsilon; 1-\epsilon; -\frac{u}{s} \right) \right], \quad (\text{B.1})$$

$$F_4^{1m}(P^2; s, t) = \frac{2}{\epsilon^2} \left[ \left( \frac{\mu^2}{-s} \right)^\epsilon {}_2F_1 \left( 1, -\epsilon; 1-\epsilon; -\frac{u}{t} \right) + \left( \frac{\mu^2}{-t} \right)^\epsilon {}_2F_1 \left( 1, -\epsilon; 1-\epsilon; -\frac{u}{s} \right) - \left( \frac{\mu^2}{-P^2} \right)^\epsilon {}_2F_1 \left( 1, -\epsilon; 1-\epsilon; -\frac{uP^2}{st} \right) \right], \quad (\text{B.2})$$

$$F_4^{2me}(P^2, Q^2; s, t) = \frac{2}{\epsilon^2} \left[ \left( \frac{\mu^2}{-s} \right)^\epsilon {}_2F_1 \left( 1, -\epsilon; 1-\epsilon; \frac{us}{P^2Q^2-st} \right) + \left( \frac{\mu^2}{-t} \right)^\epsilon {}_2F_1 \left( 1, -\epsilon; 1-\epsilon; \frac{ut}{P^2Q^2-st} \right) - \left( \frac{\mu^2}{-P^2} \right)^\epsilon {}_2F_1 \left( 1, -\epsilon; 1-\epsilon; \frac{uP^2}{P^2Q^2-st} \right) - \left( \frac{\mu^2}{-Q^2} \right)^\epsilon {}_2F_1 \left( 1, -\epsilon; 1-\epsilon; \frac{uQ^2}{P^2Q^2-st} \right) \right], \quad (\text{B.3})$$

$$F_3^{1m}(s) = \frac{1}{\epsilon^2} \left( \frac{\mu^2}{-s} \right)^\epsilon, \quad (\text{B.4})$$

$$\text{Bub}(s) = \frac{1}{\epsilon(1-2\epsilon)} \left( \frac{\mu^2}{-s} \right)^\epsilon. \quad (\text{B.5})$$

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